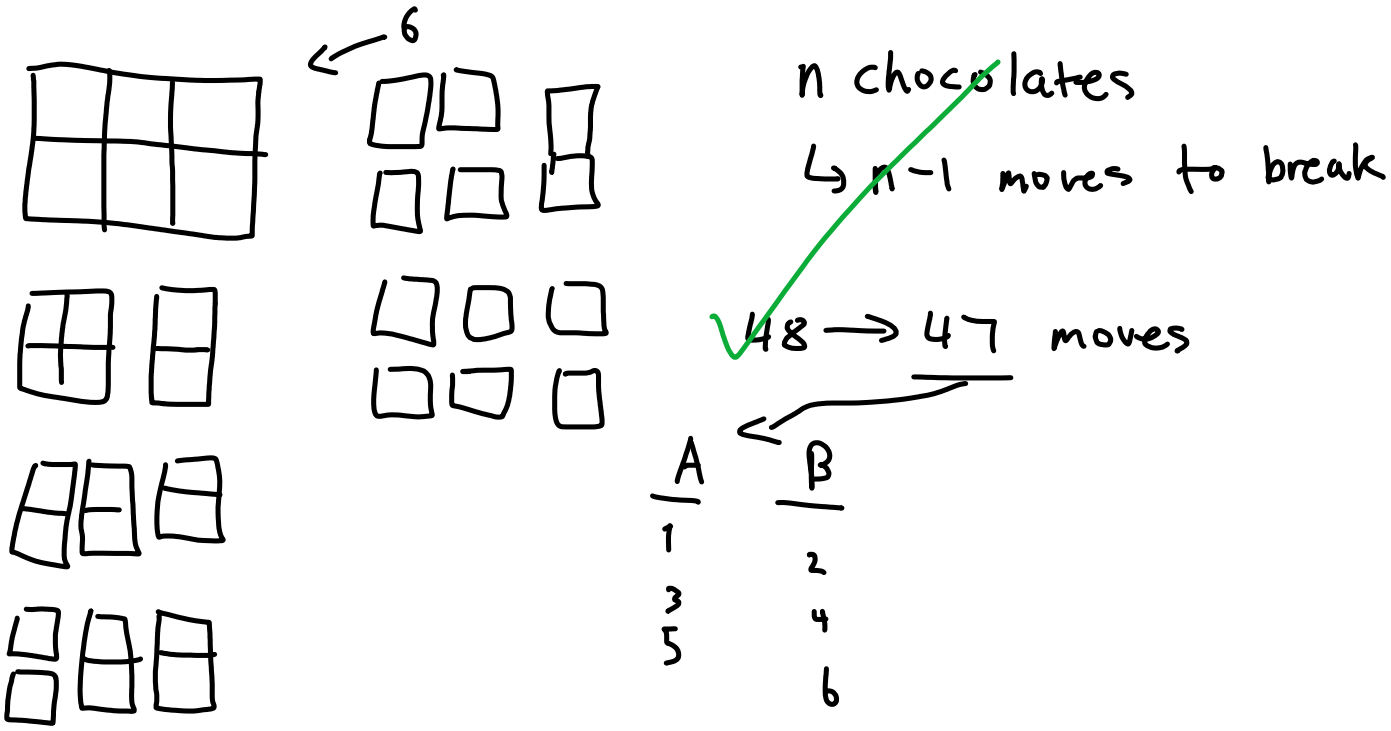
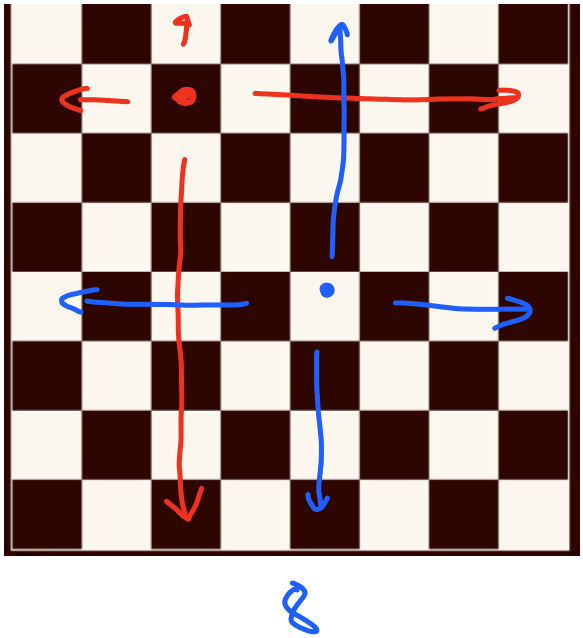
Maths Circle - Games (Dec 8)

Sunday, December 8, 2024 8:24 AM

Problem 1: Two children take turns breaking up a rectangular chocolate bar 6 squares wide by 8 squares long. They may break the bar only along the divisions between the squares. If the bar breaks into several pieces, they keep breaking the pieces up until only the individual squares remain. The player who cannot make a break loses the game. Who will win?

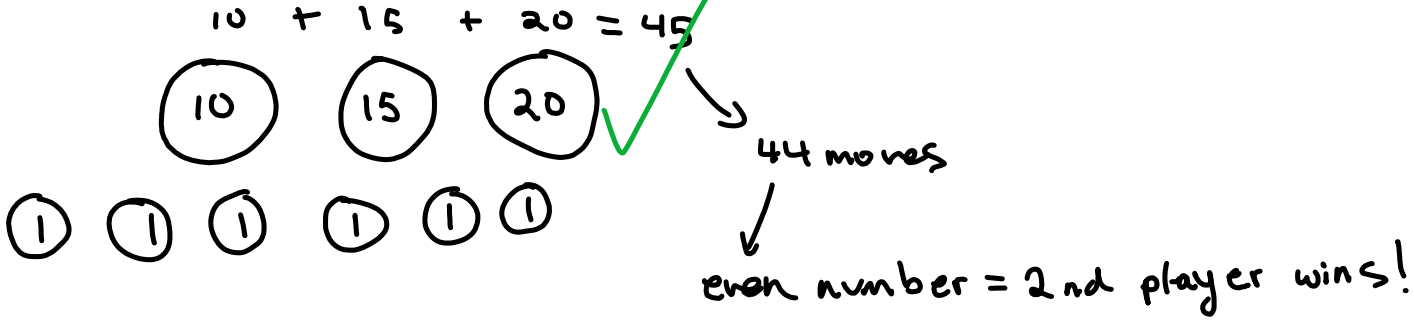


Problem 2: Two players take turns placing rooks (castles) on a chessboard so that they cannot capture each other. The loser is the player who cannot place a castle. Who will win?



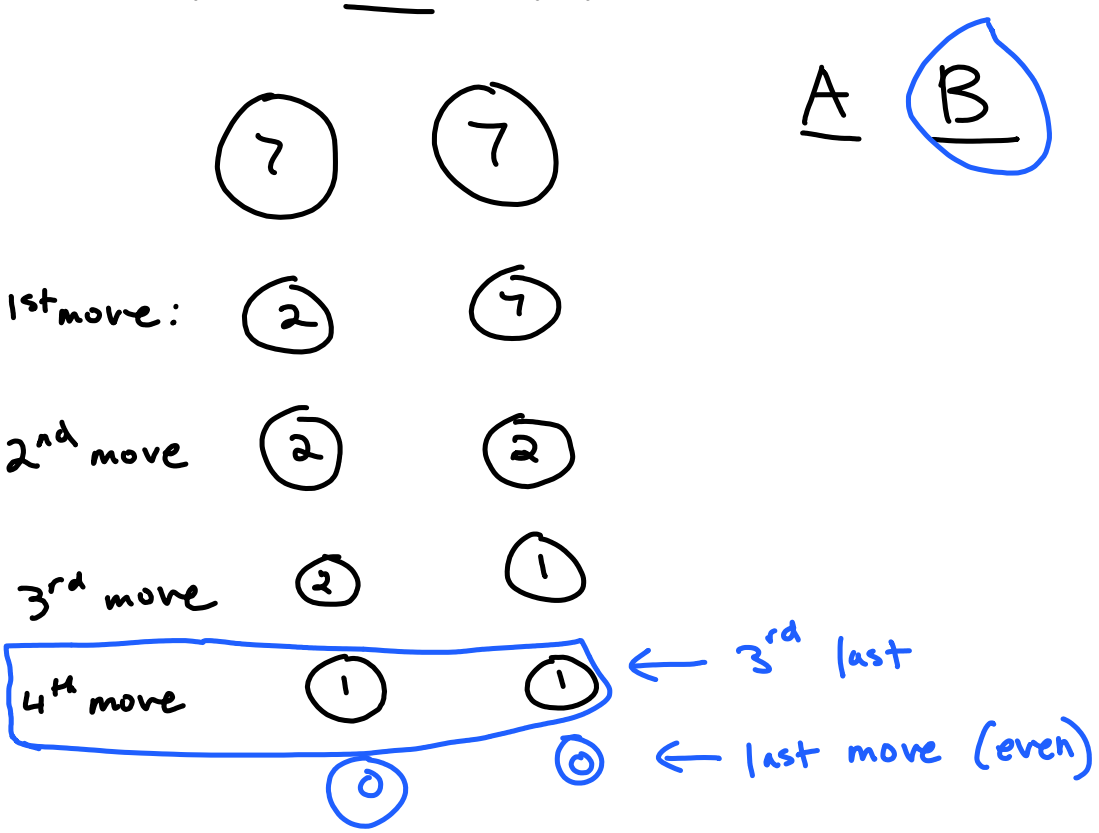
2nd player

Problem 3: There are three piles of stones: one with 10 stones, one with 15 stones, and one with 20 stones. At each turn, a player can choose one of the piles and divide it into two smaller piles. The loser is the player who cannot do this. Who will win, and how?

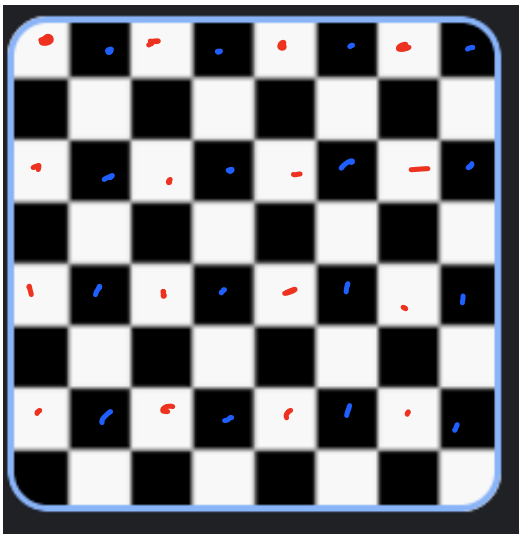


Problem 4: There are two piles of 7 stones each. At each turn, a player may take as many stones as he chooses, but only from one

of the piles. The loser is the player who cannot move



Problem 5: Two players take turns placing kings on the squares of a 8x8 chessboard, so that no king can capture another. The player who is unable to do this loses. (maximum number of moves)



16 squares

A loses (first player)

Problem 6: Two players take turns breaking a piece of chocolate consisting of 5 x 10 small squares. At each turn, they may break along the division lines of the squares. The player who first obtains a single square of chocolate wins.

50 squares → 49 moves

second player will lose

Homework:

Games Problem 1: A king is placed on square al of a chessboard. Players take turns moving the either upwards, to the right, or along a diagonal going upwards and to the right. The player wh the king on square h8 is the winner.

Games Problem 2: There are two piles of candy. One contains 20 pieces, and the other 21. Play turns eating all the candy in one pile, and separating the remaining candy into two (not necessa equal) non-empty piles. The player who cannot move loses.

Games Problem 3: A checker is placed at each end of a strip of squares measuring 1 x 20. Playe turns moving either checker in the direction of the other, each by one or by two squares. A che cannot jump over another checker. The player who cannot move loses.

king
o places

ers take
arily

rs take
ecker

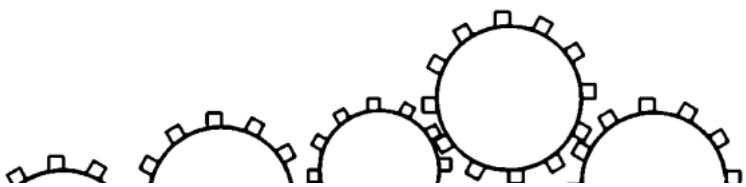
Maths Circle

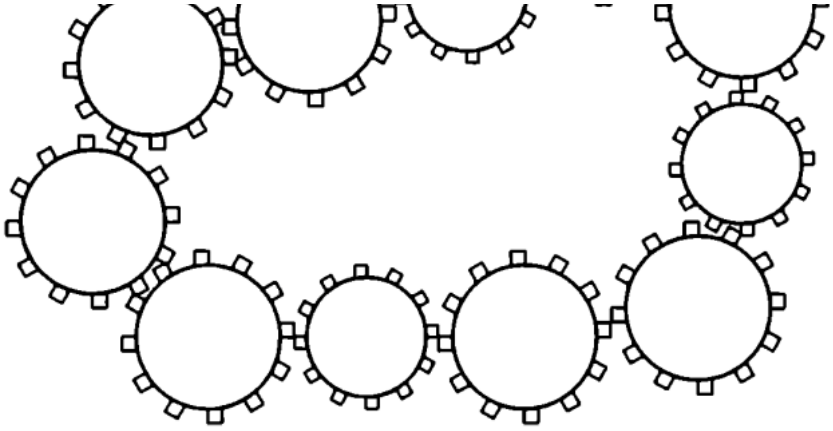
Friday, November 22, 2024 5:58 PM

What is Parity?

Parity is …

Example 1: The figure below shows a set of eleven gears arranged in a chain.





Example 2: Can a 5 x 5 square checkerboard be covered by 1 x 2 dominoes?

Example 3: Can one make change of 25 rubles, using in all ten bills each having a value of 1, 3, o
rubles?

or 5

Example 4: Katya and her friends stand in a circle. It turns out that both neighbors of each child
the same gender. If there are five boys in the circle, how many girls are there?

Example 5: Can a knight start at square a1 of a chessboard, and go to square h8, visiting each of t
remaining squares exactly once on the way?

Challenge Problem: Can an ordinary 8 x 8 chessboard be covered with 1 x 2 dominoes so that or
squares a1 and h8 remain uncovered?

are of

the

nly

Maths Circle

Friday, November 22, 2024 5:58 PM

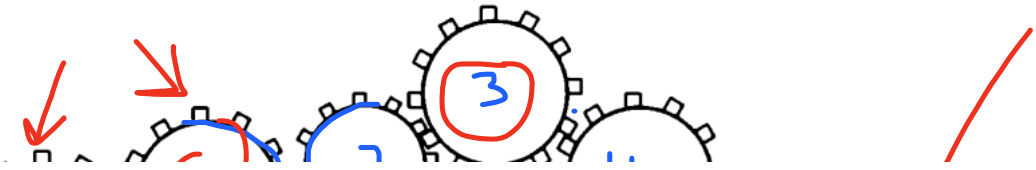
What is Parity?

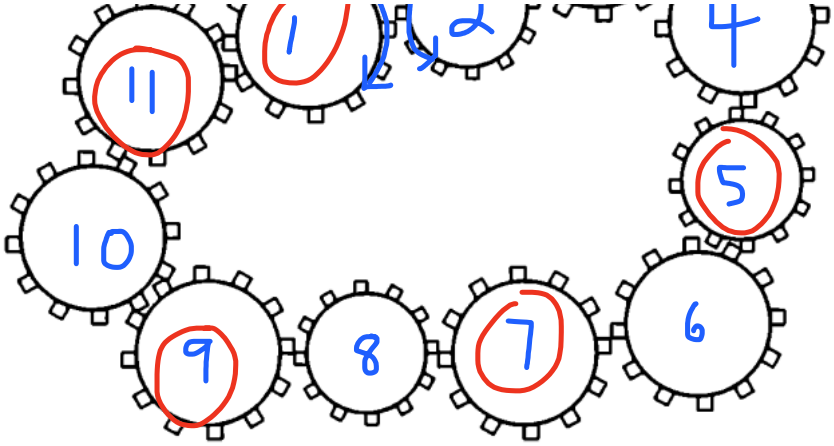
Parity is term that used to describe whether a number is even or odd.

Even number is divisible by 2, odd number is not divisible by 2

- Parity of 1? Odd parity
- Parity 32? Even parity
- 1923? Odd parity
- -56? Even parity
- 0?even parity

Example 1: The figure below shows a set of eleven gears arranged in a chain. Can all of the gears rotate simultaneously?





Example 2: Can a 5 x 5 square checkerboard be covered by 1 x 2 dominoes?

5 X 5 = 25 → odd purity

Example 3: Can one make change of 25 rubles, using in all ten bills each having a value of 1, 3, o
rubles?

even

odd

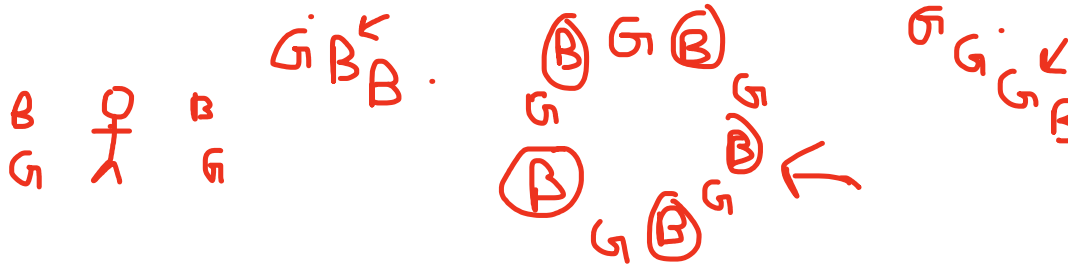
$25

$1 $3 $5 → total = 10

odd x even = even

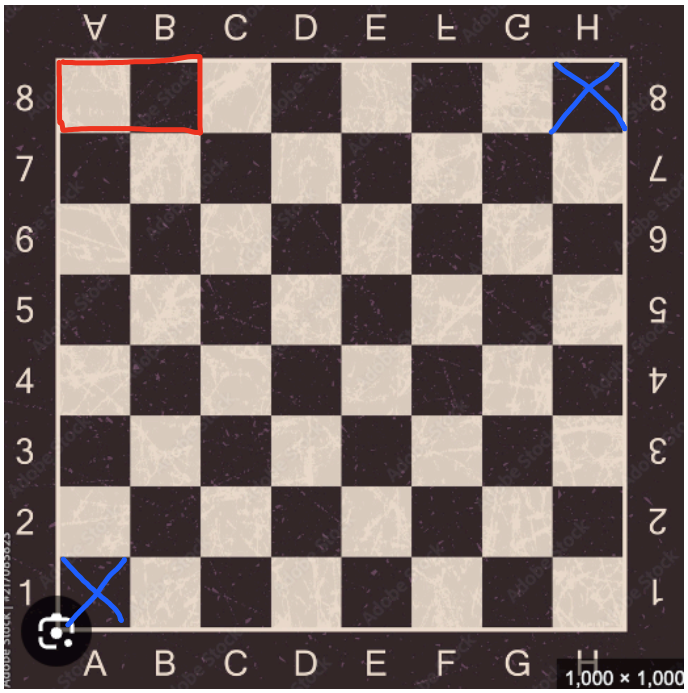
or 5

Example 4: Katya and her friends stand in a circle. It turns out that both neighbors of each child the same gender. If there are five boys in the circle, how many girls are there?



Example 5: Can a knight start at square a1 of a chessboard, and go to square h8, visiting each of t remaining squares exactly once on the way?

Challenge Problem: Can an ordinary 8 x 8 chessboard be covered with 1 x 2 dominoes so that or squares a1 and h8 remain uncovered?



Hint: think about the color of the Squares that are removed

32 white

30 black

Homework:

are of

the

nly

Parity Problem 1: A closed path is made up of 11 line segments. Can one line, not containing a vertex of the path, intersect each of its segments?

Parity Problem 2: Can we draw a closed path made up of 9 line segments, each of which intersects exactly one of the other segments?

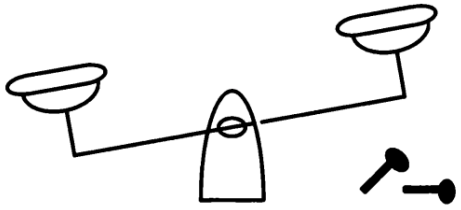
Parity Problem 3. Three hockey pucks, A, B, and C, lie on a playing field. A hockey player hits one of them in such a way that it passes between the other two. He does this 25 times. Can he return the three pucks to their starting points?

Miscellaneous Problem 1: A number of bacteria are placed in a glass. One second later each bacterium divides in two, the next second each of the resulting bacteria divides in two again, et cetera. After one minute the glass is full. When was the glass half-full?

Miscellaneous Problem 2: Jack tore out several successive pages from a book. The number of the first page he tore out was 183, and it is known that the number of the last page is written with the same digits in some order. How many pages did Jack tear out of the book?

Miscellaneous Problem 3: A caterpillar crawls up a pole 75 inches high, starting from the ground. Each day it crawls up 5 inches, and each night it slides down 4 inches. When will it first reach the top of the pole?

Miscellaneous Problem 4: There are 24 pounds of nails in a sack.
Can you measure out 9 pounds of nails using only a balance with
two pans?



Maths Circle Junior - Algebra

Friday, August 1, 2025 11:09 PM

Problem 1) How many positive integers smaller than 400 can you get as a sum of 11 consecutive positive integers?

$1+2+\dots+11=66$

$2+3+\dots+12=77$

multiples of 11 from 66 to 396

66, 77, ... , 396

↓ ↓ ↓ ↓

6 7 8 36

$36-6+1=\boxed{31}$

Problem 2) Sequence a_n is generated by the rule, $a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$.

Given $a_1 = 2$ and $a_2 = 4$, find the sum of the first 2019 terms.

$a_1 = 2$

$a_2 = 4$

$a_3 = 4-2 = 2$

$a_4 = 2-4 = -2$

$a_5 = -4$

$a_6 = -2$

$a_7 = 2$

$a_8 = 4$

2, 4, 2, -2, -4, -2, 2, 4, 2, -2, -4, -2

(0) (0)

$2019 = 2016 + 3$

$= (336 \times 6) + 3$

336

0 0 0 0 0 2, 4, 2

$2+4+2=\boxed{8}$

Problem 3: Given $a,b,c \in \mathbb{R}$ and $9a+b+8c=12$ and $8a+12b+9c=1$.

$a^2-b^2+c^2=?$

$9a+8c = 12-b$ ----- ①

$8a-9c = 1-12b$ ----- ②

$①^2 + ②^2$

$(9a+8c)^2 + (8a-9c)^2 = (12-b)^2 + (1-12b)^2$

$145(a^2+c^2) = 145(b^2+1)$

$a^2-b^2+c^2 = \boxed{1}$

Problem 4) $a,b,c,d \in \mathbb{Z}^+$, $a + \cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{43}{30}$, $d=?$

$\frac{43}{30}$

$= 1 + \frac{13}{30}$

$= 1 + \cfrac{1}{\frac{30}{13}}$

$= 1 + \cfrac{1}{2+\frac{4}{13}}$

$= 1 + \cfrac{1}{2+\cfrac{1}{\frac{13}{4}}}$

$= 1 + \cfrac{1}{2+\cfrac{1}{3+\frac{1}{4}}}$

a b c d

Maths Circle Junior - Algebra

Friday, July 4, 2025 11:21 PM

Problem 1:
Solve for x: (x + 3)(x - 2) = (x + 1)(x - 4)

$(x+3)(x-2)-(x+1)(x-4)=0$

$x^2+x-6-(x^2-3x-4)=0$

$4x-2=0$

$x^2+x-6=x^2-3x-4$

$4x=2$

$x=\frac{1}{2}$

Problem 2:
If a + b = 5 and ab = 6, find $a^2 + b^2$.

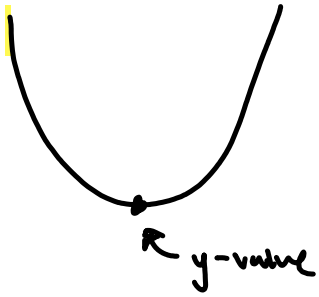
$(a+b)^2=5^2$

$a^2+2ab+b^2=5^2$

$a^2+2ab+b^2-2ab=5^2-12$

$a^2+b^2=13$

Problem 3:
Let $f(x) = x^2 - 4x + 7$. Find the minimum value of $f(x)$.



$f(x) = a(x-h)(x-k) + d$

$f(x) = a(x+h)^2 + k$

(h, k)

$f(x) = x^2 - 4x + 4 + 3$

$= (x-2)^2 + 3$

$(x-2)^2$

$= x^2 - 4x + 4$

$a = 1$

$h = 2$

$k = 3$

$(2, 3)$

minimum value = 3

Problem 4:
Solve for real x: $2x + \sqrt{(x + 5)} = 10$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\sqrt{x+5} = 10-2x$$

$$x+5 = (10-2x)^2$$

$$x+5 = 4x^2-40x+100$$

$$4x^2-41x+95 = 0$$

$$x = \frac{41 \pm \sqrt{161}}{8}$$

Problem 5:
Let x and y be real numbers such that $x + y = 4$ and $x^2 + y^2 = 10$. Find the value of xy.

Problem 6:
If $3^{(x + 1)} + 3^{(x)} = 108$, find the value of x.

Problem 7:
Let $f(x) = 1 / (x + 1)$. Find $f(f(f(1)))$.

Problem 8:
Find all real solutions to the equation $x^4 - 5x^2 + 4 = 0$.

Problem 9:
The sum of two numbers is 12, and the difference of their squares is 48. Find the two numbers.

Problem 10:
If a and b are real numbers such that $a + b = 3$ and $a^3 + b^3 = 63$, find the value of ab.

Maths Circle Junior - Algebra

Friday, March 28, 2025 9:16 PM

$5x^2$
$a=5$

1. Quadratic Equation Roots

The roots of the quadratic equation $x^2-5x+k = 0$ are in the ratio **2:3**. Find the value of k.

$\alpha : \beta$

$\alpha = 2m$
$\beta = 3m$

x^2-5x+k
$a = 1$
$b = -5$
$c = k$

Vieta formula:

$y = ax^2 + bx + c \rightarrow$ roots α and β

$\alpha + \beta = -\frac{b}{a}$

$\alpha\beta = \frac{c}{a}$

$\alpha \rightarrow 2m$
$\beta \rightarrow 3m$

$2m + 3m = 5m = -\frac{-5}{1} = 5$

$5m = 5$

$m = 1$

$\frac{c}{a} = \frac{k}{1} = k = (2)(3) = 6$

2. Exponential Growth

A bacteria colony doubles in size every 3 hours. If it starts with 500 bacteria, how many will there be after 12 hours?

$500 \rightarrow 1000 \rightarrow 2000 \rightarrow 4000 \rightarrow 8000$

$\times 2 \quad \times 2$

$\frac{12}{3} = 4$ doubling times

500×2^4

3. System of Equations

Solve for x + y if:

$2x + 3y = 1$ ← ×2 → $4x + 6y = 2$

$4x - y = 5$ → $-(4x - y = 5)$

$0 \quad 7y = -3$

$y = -\frac{3}{7}$

$x + y = \frac{8}{7} - \frac{3}{7} = \boxed{\frac{5}{7}}$

$4x + \frac{3}{7} = 5$

$x = \frac{8}{7}$

4. Sum of a Sequence

Find the sum of the arithmetic sequence:

2,5,8,…,50

$\frac{n}{2}(2a + (n-1)d)$

$= \frac{17}{2}(2(2) + (17-1)(3))$

$= 442$

$\frac{n}{2}(a + a_n)$

$= \frac{17}{2}(2 + 50)$

$= 442$

$a + (n-1)d = a_n$

$2 + (n-1)(3) = 50$

$n = 17$

$n \rightarrow$ number of terms

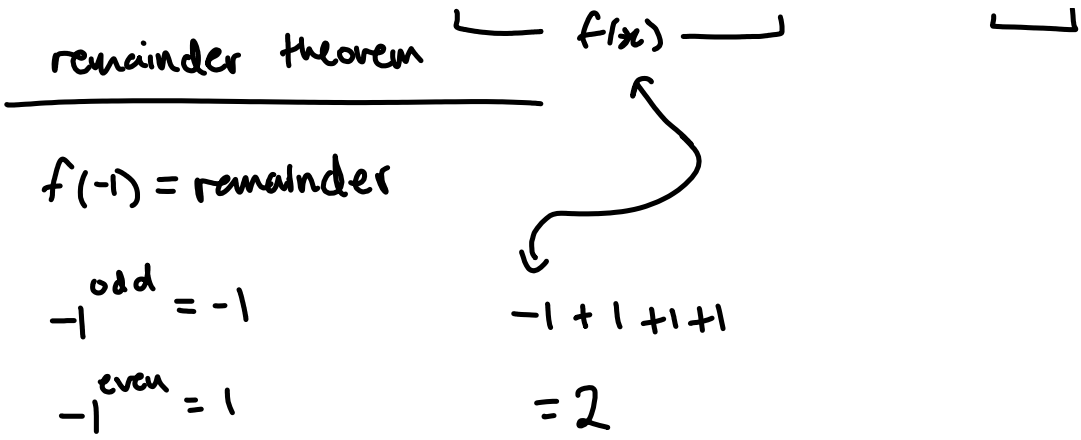
$a \rightarrow$ first term

$d \rightarrow$ common difference

$a_n \rightarrow n^{th}$ term

5. Polynomial Remainder

Find the remainder when $x^{2025} + x^{1012} + x^{506} + 1$ is divided by $x + 1$.

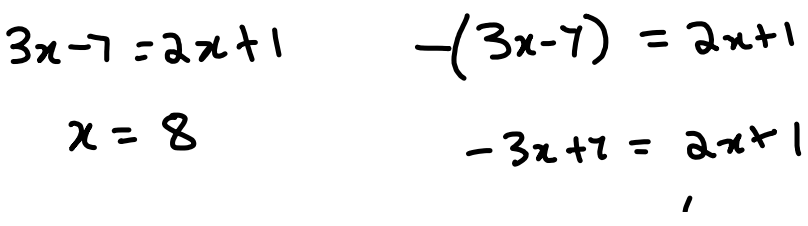


6. Functional Equation

A function f satisfies:
f(x + y)=f(x) + f(y) + 2xy for all real numbers x and y, and it is given that f(1)=3. Find f(5).

7. Absolute Value Equation

Solve for x:
|3x−7|=2x+1



$x = \frac{6}{5}$

8. Geometric Sequence

The first three terms of a geometric sequence are **a, b, and c**, where
a + b + c = 21 and a, b, c form a geometric sequence with common ratio r.
If a = 3, find c.

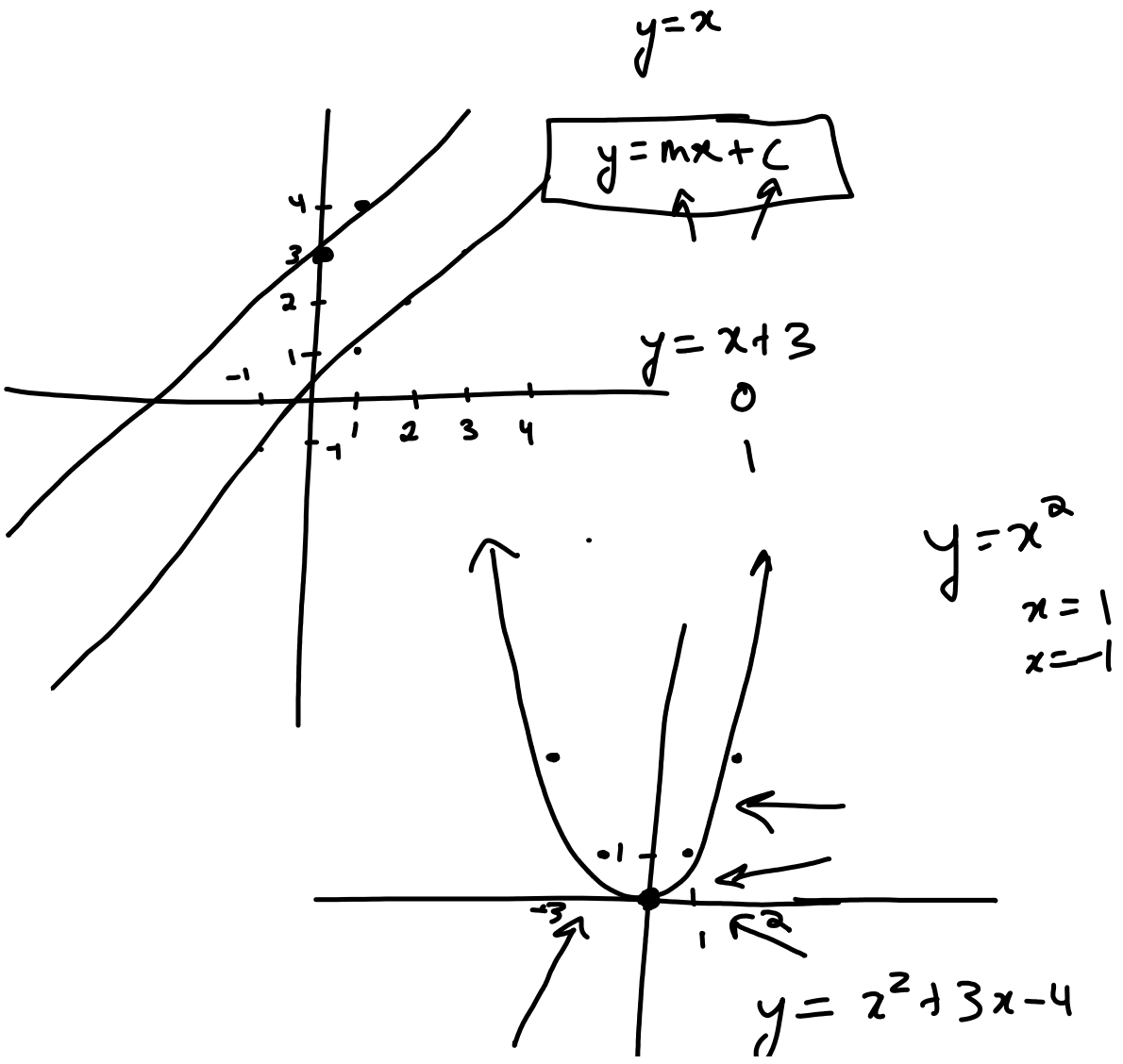
9. Vieta's Formulas

The roots of the equation $x^2-7x+12=0$ are p and q. Find the value of p^2+q^2

10. Quadratic Discriminant

For what values of k does the equation $2x^2-4x+k=0$ have real and distinct roots?

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$



Maths Circle Junior - Algebraic Formulas

Friday, July 11, 2025 11:16 PM

Problem 1
Evaluate: $(7 + 3)^2 - (7 - 3)^2$

$= (10)^2 - (4)^2$
$= 100 - 16$
$= 84$

$(a+b)^2 - (a-b)^2 = 4ab$

Problem 2:
If $a + b = 10$ and $ab = 21$, find $a^2 + b^2$.

a^2+b^2
$= (a+b)^2 - 2ab$
$= 10^2 - 2(21)$
$= 100 - 42$
$= 58$

Problem 3:
Factor completely: $x^3 - 27$

$(a+b)^3 = a^3+b^3+3ab^2+3a^2b$
$(a-b)^3 = a^3-b^3-3ab^2+3a^2b$
$a^3+b^3 = (a+b)(a^2-ab+b^2)$
$a^3-b^3 = (a-b)(a^2+ab+b^2)$
$\hookrightarrow \boxed{(x-3)(x^2+3x+9)}$

Problem 4:
If $x^2 - 4x + 5 = 0$, find the value of $x^4 + 1$.

Problem 5:
Let a and b be real numbers such that $a^2 + b^2 = 20$ and $ab = 6$. Find $(a + b)^2$.

$(a+b)^2$
$= a^2+b^2+2ab$
$= 20 + 2(6)$
$= 32$

Problem 6:
Given that x and y satisfy $x + y = 8$ and $x^2 + y^2 = 50$, find the value of xy.

$2xy = (x+y)^2 - (x^2+y^2)$
$2xy = 8^2 - 50$
$2xy = 14$
$\boxed{xy = 7}$

Problem 7:
Simplify: $(x^2 - 9) / (x - 3)$

* rational expressions *

$\frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{x-3} = x+3, x \neq 3$

Problem 8:
Let $x + 1/x = 3$. Find $x^2 + 1/x^2$.

$x + \frac{1}{x} = 3$

$\left(x+\frac{1}{x}\right)^2 = 3^2$

$x^2 + 2 + \frac{1}{x^2} = 9$

$\boxed{x^2+\frac{1}{x^2} = 7}$

Problem 9:
If r and s are the roots of the equation $x^2 - 7x + 12 = 0$, find $r^2 + s^2$.

Vieta's Theorem for quadratic: α, β of $ax^2+bx+c=0$

$\alpha+\beta = -\frac{b}{a}$
$\alpha\beta = \frac{c}{a}$

$r=4$
$s=3$ → factoring or quadratic

$x^2-7x+12=0$
$r+s = -\frac{-7}{1} = 7$
$rs = \frac{12}{1} = 12$
$r^2+s^2 = (r+s)^2 - 2rs$
$= 7^2 - 2(12)$
$= \boxed{25}$

Problem 10:
Factor and solve: $x^3 + x^2 - 4x - 4 = 0$ ($x=2$, $x=-1$, $x=-2$)

$x^2(x+1) - 4(x+1) = 0$
$(x+1)(x^2-4) = 0$
$(x+1)(x+2)(x-2) = 0$
$x = -1, -2, 2$

Problem 11
$\frac{2x^2-8}{x^2-4}$

Simplify the expression.

$2\frac{(x^2-4)}{(x^2-4)} = 2, x \neq 2, -2$

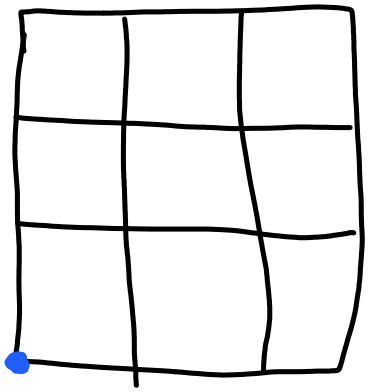
Maths Circle Junior - Combinatorics

Saturday, March 22, 2025 12:00 AM

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Problem 1:

A robot starts at the bottom-left corner of a 3×3 grid and wants to reach the top-right corner. It can only move right or up. How many different paths can the robot take?



3 right

3 up

total paths = ${}^{6}C_{3} = \frac{}{3}$

3R 3U

Problem 2:

At a party, there are 8 people, and each person shakes hands with every other person exactly once. How many total handshakes occur?

$$\frac{n(n-3)}{2}$$

$x^5 y^4$

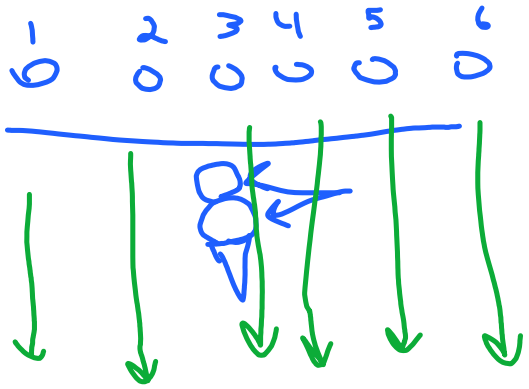
$$\frac{6!}{3!} = 20$$

Problem 3:

An ice cream shop offers 6 flavors. You want to buy a double scoop cone, where both scoops can be the same or different flavors. How many different double scoop cones can you order?

n

r



$6 + 5 + 4 + 3 + 2 + 1 = 21$

$$n+r-1C_r = {}^{6+2-1}C_2 = {}^{7}C_2 = \frac{7!}{2!(7-2)!} = \frac{7 \times 6^3 \cdot 5!}{2! \cancel{(5)!}} = 21$$

Problem 4:

How many ways can the letters in the word "MATH" be arranged?

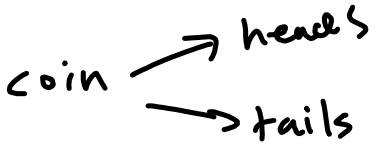
$4 \times 3 \times 2 \times 1 = 24$

M
A
T
H

Problem 5:

A fair coin is flipped 4 times. How many different sequences of heads and tails are possible?

$2 \times 2 \times 2 \times 2 = 2^4 = 16$



Problem 6:

A club has 10 members, and a 3-person committee needs to be selected. How many different ways can the committee be formed?

$10 C_3 = \frac{10 \times 9 \times 8}{10!} = 120$

3 7!(7!

Problem 7:

Four friends sit in a row of 4 chairs. If two of them insist on sitting next to each other, how many different seating arrangements ~~are possible~~?

→ one object

— — —

3! × 2! = 6×2 = 12

Problem 8:

A set contains 5 elements: {A, B, C, D, E}. How many different subsets (including the empty set and the full set) can be formed?

formed?

n is # of elements

$2^n = 2^5 = 32$

Problem 9:

A license plate consists of 3 letters followed by 2 digits. If the letters can be any of the 26 English letters and the digits can be any of the 10 digits (0-9), how many different license plates are possible?

$\underset{L}{26} \times \underset{L}{26} \times \underset{L}{26} \times \underset{D}{10} \times \underset{D}{10} = 26^3 \times 10^2 = \boxed{17576000}$

Problem 10:

A birthday cake has 7 candles, and each candle can be either blue, red, or yellow. How many different ways can the candles be colored?

$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^7 = \boxed{2187}$

Maths Circle Junior - Divisibility and Remainders

Friday, January 3, 2025 10:53 PM

What is a prime number?

number which has 2 divisors: 1 and itself

What is a composite number?

number which has more than 2 divisors

Fundamental Theorem of Arithmetic:
Any natural number different from 1 (2,3,4,5….) can be written as the product of prime number in increasing order

15

$1 \times 3 \times 5$

18

2×9

$2 \times 3 \times 3$

What is relatively prime, or coprime?

HCF (A, B) = 1

GCD

A and B have no common factors

What is factorial?

$n! = n \times (n-1) \times (n-2) \times \ldots 1$

$3! = 3 \times 2 \times 1$

Problem 1: If a number A is even, is it true that 3A must be divisible by 6?

even → $2k = A$

$3A = 2k \times 3 = 6k$

Problem 2: Given the numbers $A = 2^8 \cdot 5^3 \cdot 7$ and $B = 2^5 \cdot 3 \cdot 5^7$ find LCM(A, B) Lowest Common Multiple

$2^8 \times 5^7 \times 7 \times 3$

A: $2^8 \times 5^3 \times 7^1$

B: $2^5 \times 3^1 \times 5^7$

$2^8 \times 3 \times 7 \times 5^7$

Problem 3: Prove that the product of any three consecutive natural numbers is divisible by 6.

$6 = 3 \times 2$ ✓

x, y, z

1, 2, 3, 4, 5, 6....

Problem 4: Find the smallest natural number n such that n! is divisible by 990.

$n! = n \times (n-1) \times \cdots \times 2 \times 1$

$\frac{n!}{990} = 2$

$\frac{990}{n!} = 2$

$11 \times 10 \times 9 \times 8 \times 7$

$990 = 2 \times 3^2 \times 5 \times 11$

$n! \rightarrow 2, 3^2, 5, 11$

$11! = 11 \times 10 \times 9 \times 8 \times \cdots 1$

Problem 5: Find all solutions in natural numbers of the equations

a) $x^2 - y^2 = 31$

$31 = 1 \times 31$

$(x+y)(x-y) = 31$

$x+y = 31$

$x-y = 1$

$x+y = 1$ | $2x = 32$

$x-y = 31$

$x = 16$

$y = 15$

$x^2 - y^2$

$= (x+y)(x-y)$

(c) $x^2 - y^2 = 148$

$(x+y)(x-y)$

b) $x^2 - y^2 = 303$.

$303 = 3 \times 101$

$(x+y)(x-y) = 3 \times 101$

$x+y = 101$

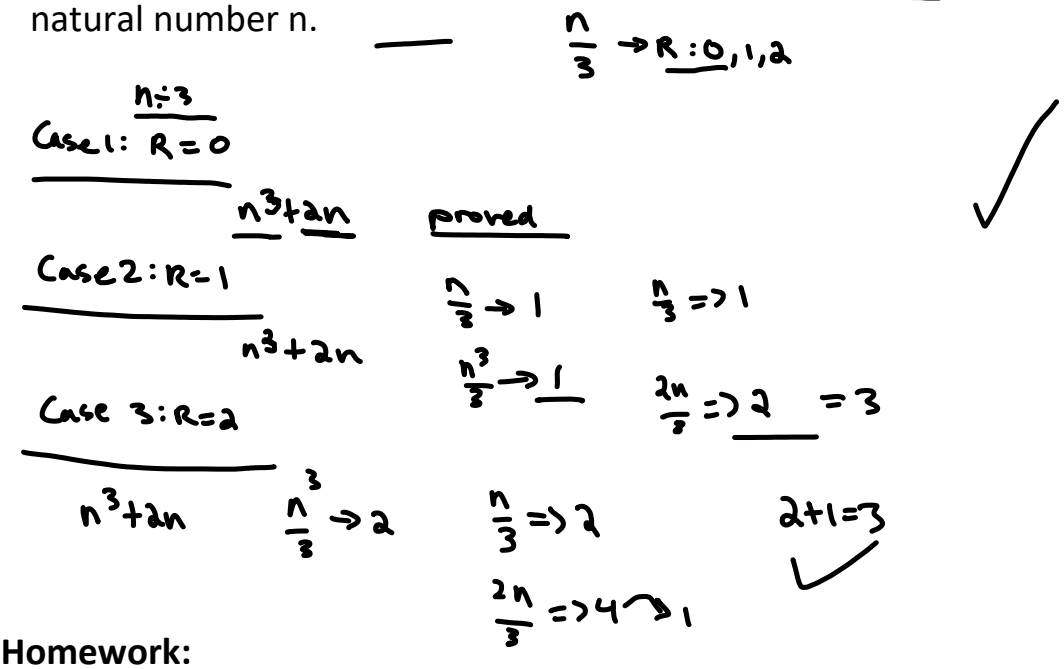
$x-y = 3$

$2x = 104$

$x = 52$

$y = 49$

Problem 6: Prove that the number $n^3 + 2n$ is divisible by 3 for any natural number n.



Homework:

Divisibility Problem 1: Prove that any two natural numbers a and b satisfy the equation GCD(a, b) LCM(a, b) =ab.

Divisibility Problem 2: The numbers a and b satisfy the equation 56a = 65b. Prove that a + b is composite.

Divisibility Problem 3: How many zeros are there at the end of the decimal representation of the number 100!

Miscellaneous Problem 1: Bindu's bike tires have a radius of 30 cm. She rides her bike far enough that the tires rotate exactly five times. How far does Bindu's bike travel?

Miscellaneous Problem 2: For 30 consecutive days, the daily high temperature was recorded. On each of the first 25 days, the temperature recorded was 21℃. On each of the remaining 5 days, the temperature recorded was 15℃. For the 30 days, the mean (average) of the temperatures recorded was

Miscellaneous Problem 3: The product of a pair of 2-digit positive integers is 630. How many such pairs are there?

Maths Circle Junior - Divisibility

Friday, April 25, 2025 11:33 PM

Problem 1: How many positive integers less than 1000 are divisible by either 6 or 10 but not both?

divisible by 6 → 166 numbers

divisible by 10 → 99 numbers

divisible by LCM(10,6) → 33 numbers
"30"

166 + 99 − (33)(2) = **199 numbers**

Problem 2: Find the greatest three-digit number that leaves a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, and a remainder of 3 when divided by 4.

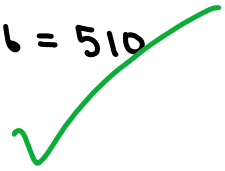
$-1 \bmod 12 \rightarrow 11 \bmod 12$

$(82 \times 12) + 11 = 995$

$\frac{1000}{12} \rightarrow 83$

Problem 3: What is the smallest positive integer such that when divided by 7, 8, and 9, the remainder is 6 each time?

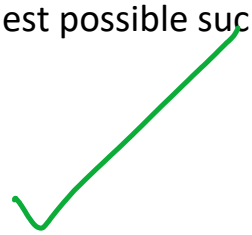
LCM (7,8,9) → 504 + 6 = 510



Problem 4: A five-digit number ends in 5 and is divisible by 15 and 9. What is the smallest possible such number?

LCM (15,9) → 45

$\frac{10000}{45} \div 222, r = 10$



45 × 223 = 10035

Problem 5: How many four-digit positive integers are divisible by 36?

1000 → 9999 → 9000 integers

$\frac{9000}{36} = \boxed{250 \text{ integers}}$

Problem 6: Find the smallest positive integer n such that n is divisible by 5, and n + 1 is divisible by 6, and n + 2 is divisible by 7.

$n > 5$

215

Problem 7: How many integers between 1 and 1000 are divisible by neither 4 nor 6?

667 ✓

Problem 8: What is the sum of all positive integers less than 100 that are divisible by both 3 and 7?

21+42+63+84 ✓

= 210

Problem 9: How many three-digit numbers are divisible by 8 and have all digits distinct?

Problem 10: Let N be the smallest positive number such that when multiplied by 9, the product is a number consisting only of the digit 1. Find N.

Maths Circle Junior - Functions (Algebra)

Saturday, January 25, 2025 9:23 AM

$f(x) = 5x+3$

$f(5) = 28$ →

What is a function?

operation that takes in one value and gives **one** output value

$f(x) = x^2+4$, $f(4) = 20$, $f(7) = 53$

$f(4) = 4^2+4$

$4 \rightarrow [x^2+4] \rightarrow 20$

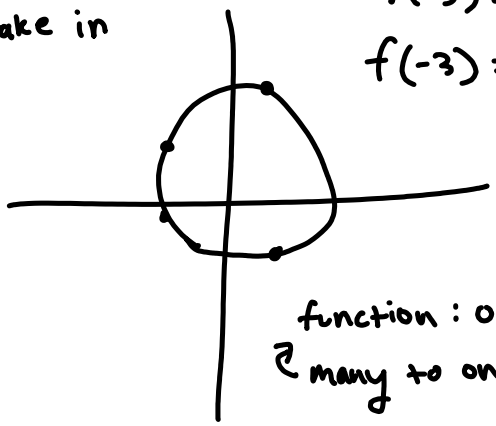
$f(3) = 3^2+4 = 13$

$f(-3) = (-3)^2+4 = 13$

What is the domain of a function?

all values that the function can take in

$D: x \in \mathbb{R}$



function : one to one

many to one

one to many

What is the range of a function?

all values that a function can give

$R: x \in \mathbb{R}$

↑

x belongs to real numbers

What is the inverse of a function?

opposite of a regular function

$f(x) = x^2+4$

$y = x^2+4$

inverse → $f^{-1}(x)$

$x = y^2+4$

$y^2 = x-4 \rightarrow y = \sqrt{x-4}$

$f(x) = 5x+3$

$y = 5x+3 \rightarrow f^{-1}(x) \Rightarrow x = 5y+3$

$5y = x-3$

$y = \frac{x-3}{5}$

$x < 4$

$y = \sqrt{1-4} = \sqrt{-3} =$

$D: x \in \mathbb{R}, x \geq 4$

Problem 1: Which of the following relations are functions? Give reasons and also find the domain and range of the function.

(i) f = {(1, 3), (1, 5), (2, 3), (2, 5)} ✗

(ii) g = {(2, 1), (5, 1), (8, 1), (11, 1)} ✓

f(x)

f(1) → 3 5

many { • (1,5) • (1,3)

one

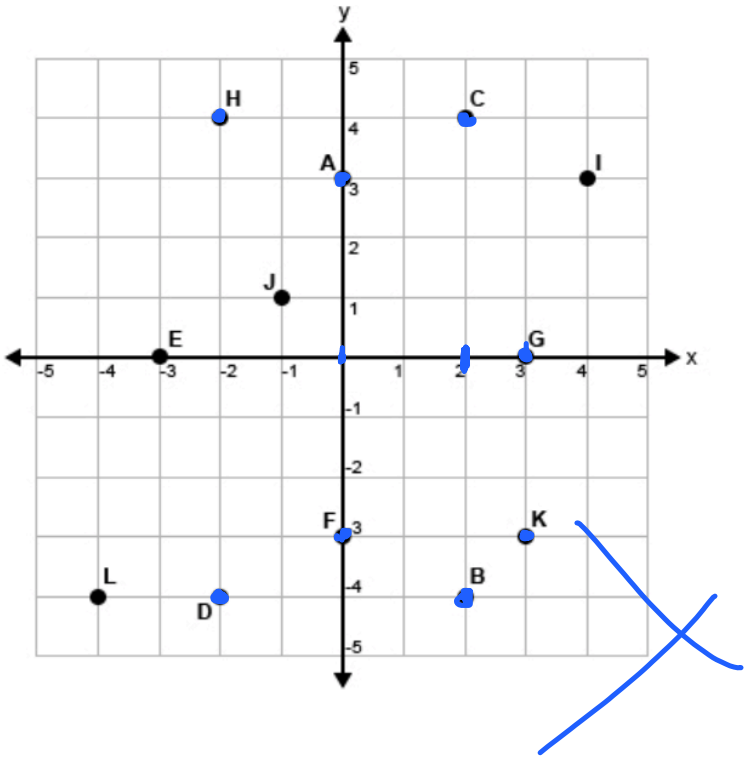
(i) D: $x \in \{1, 2\}$ R: $\{3, 5\}$

(ii) D: $x \in \{2, 5, 8, 11\}$ R: $\{1\}$

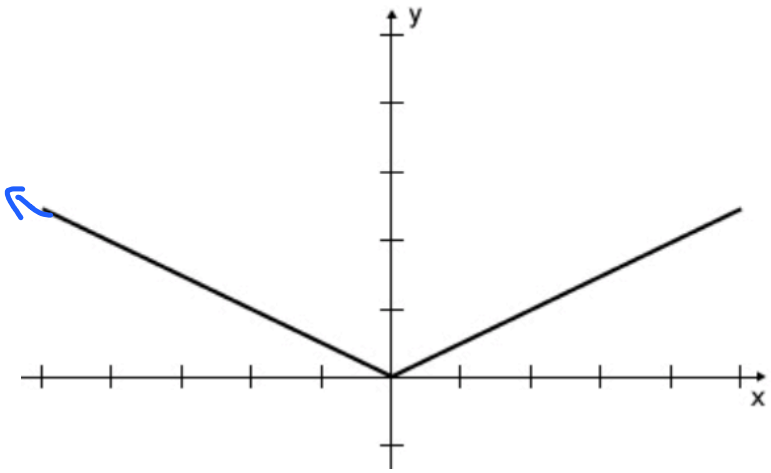
Problem 2: Let $f(x) = x^2 + 1$. Find $f(10)$.

101

Problem 3: Is the following a function?



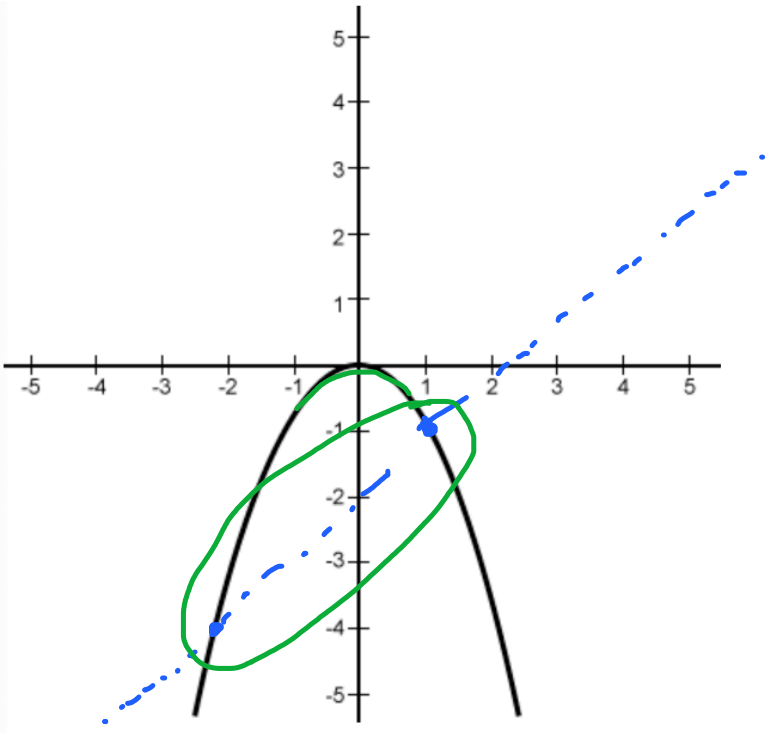
Problem 4: What is the domain and range for the function below?



D: $x \in \mathbb{R}$

R: $y \in \mathbb{R}^+, 0$

Problem 5: At what values of the domain is the function f(x) = x - 2 less than the function represented in the graph below?



$y = -x^2$

$y = x - 2$

$-x^2 = x - 2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2$ or $x = 1$

$x \in (-2, 1)$

$x \in [-2, 1]$

Problem 6: A car's value depreciates each year by 15%. Which function would best model the car's value depending on the number of years the owner has had it?

A) Linear

B) Exponential

C) Quadratic

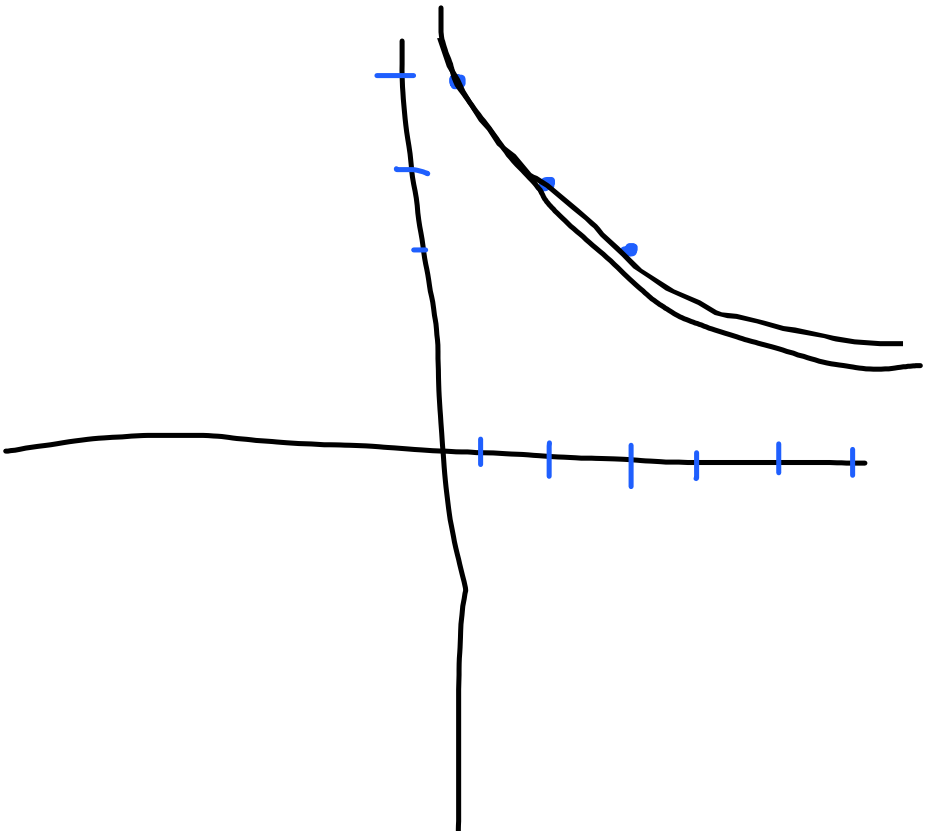
D) None of the above

$f(x) = x\left(1 - \frac{15}{100}\right)^x$

$= x(0.85)^x$

100 → 15% of 100 → 15 → 85

85 → 15% of 85 → 12.75 → 72.25



Maths Circle Junior - Games

Saturday, June 28, 2025 9:30 AM

Problem 1:
Alice and Bob take turns removing 1, 2, or 3 stones from a pile of 17. Alice goes first. The player who removes the last stone wins. Who has a winning strategy?

Alice will win

$17 \equiv 1 \mod 4$

$17 \xrightarrow[A]{} 16 \xrightarrow[B]{} \xrightarrow[A]{} 12 \xrightarrow[B]{} \xrightarrow[A]{} 8$

$4 \xrightarrow[B]{} 1,2,3 \xrightarrow[A]{} 0$

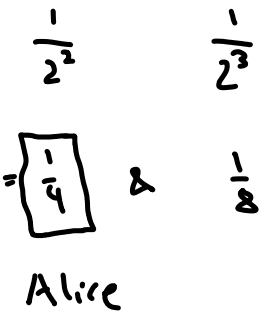
Problem 2:
Two players take turns choosing distinct integers from 1 to 9. The first player to have any three numbers that sum to 15 wins. If all numbers are chosen and no one has won, the game is a draw. Who has a winning strategy?

Problem 3:
In a two-player game, players alternate writing either an X or an O in a 3×3 grid. The first player to fill an entire row, column, or diagonal with the same letter wins. If the grid fills up with no winner, it's a draw. Can the second player force a win?

.

Problem 4:
A coin is flipped repeatedly. Alice scores a point when

two heads appear consecutively (HH). Bob scores a point when the pattern heads-tails-heads (HTH) appears. Who is more likely to score first?



Problem 5:
Two players play a game where they start at the number 1. On each turn, a player multiplies the current number by 2 or 3. The player who reaches or exceeds 100 first wins. Who has a winning strategy if the first player starts?

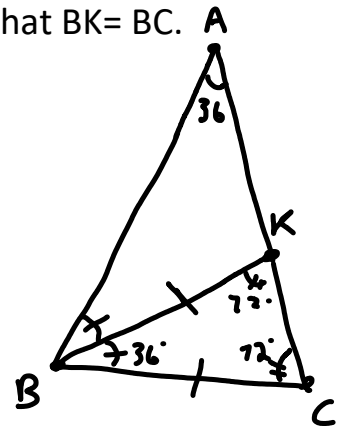
Problem 6:
A 5×5 grid is initially empty. Two players take turns placing either a black or a white stone in an empty square. A player wins if four stones of their color form a line. Is it possible for the first player to guarantee a win?

Problem 7:
Players take turns subtracting a square number (1, 4, 9, ...) from a pile of 100 coins. The player who makes the last move wins. Who has a winning strategy?

Maths Circle Junior - Geometry (Angles)

Sunday, January 19, 2025 9:22 AM

Problem 1: Angle bisector BK is drawn in isosceles triangle ABC, with angle A equal to 36 degrees. Prove that BK= BC.

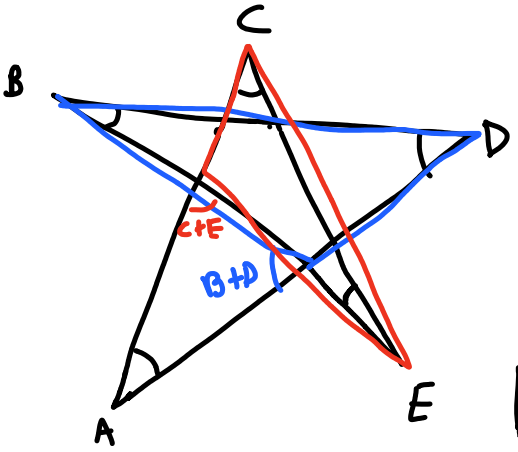


$\angle ABC = \angle ACB = 72^\circ$

$\angle ABK = \angle KBC = \frac{72}{2} = 36^\circ$

$\triangle KBC : \angle BKC = 72^\circ$.

Problem 2:Prove that the sum of the angles at the vertices of a five-pointed star equals 180 degrees.



$\angle BFE = 180 - \angle B - \angle E$

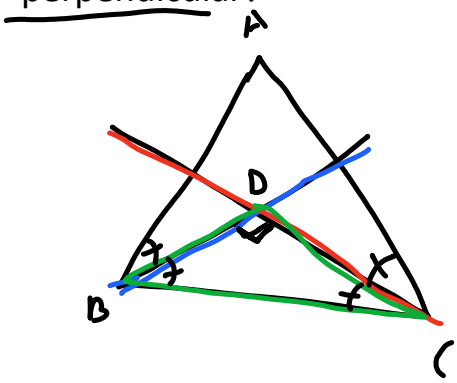
$\angle AGD = 180 - \angle A - \angle D$

$\angle CGF = 180 - \angle AGD = \angle A + \angle D$

$\angle CFG = 180 - \angle BFE = \angle B + \angle E$

$A+B+C+D+E = 180$

Problem 3: Can two angle bisectors in a triangle be perpendicular?

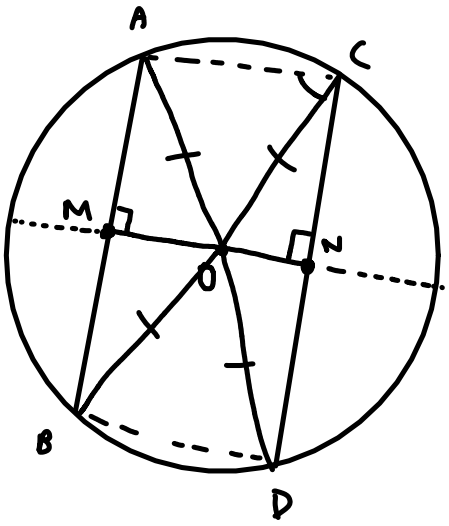


under the assumption $\angle BDC = 90^\circ$

$\angle DBC + \angle BCD = 90^\circ$

$\angle ABC + \angle ACB \neq 180$

Problem 4: Chords AB and CD in a circle are parallel. Prove that AC= BD.



Problem 5: The ratio of three consecutive angles in

an inscribed quadrilateral is 2 : 3 : 4. Find their values.

Problem 6: Square ABCD is given. A circle with radius AB and center A is drawn. This circle intersects the perpendicular bisector of BC in two points, of which O is the closest to C. Find the value of angle AOC.

Homework Problems:

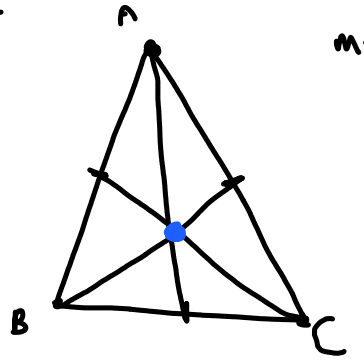
Problem 1: Cut a square into 5 rectangles in such a way that no two of them have a complete common side {but may have some parts of their sides in common).

Problem 2: Is it possible to draw a closed 8-segment broken line which intersects each segment of itself exactly once?

Maths Circle Junior - Geometry

Friday, December 13, 2024 5:24 PM

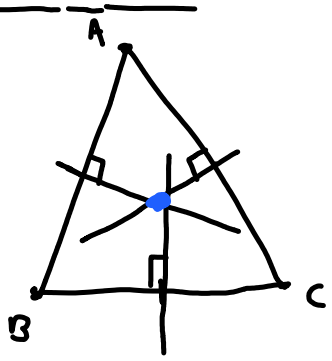
Median and centroid of a triangle:



median: vertex to midpoint of the opposite side

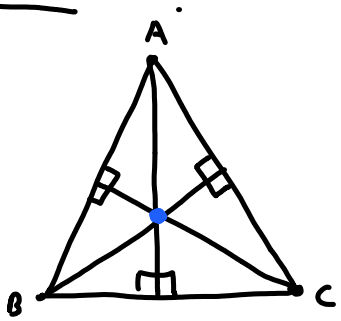
centroid: intersection of all medians

Perpendicular bisector and circumcenter:



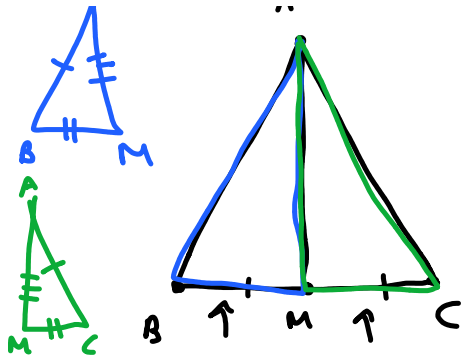
bi = 2

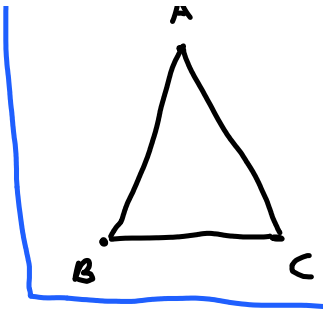
Altitude and orthocenter:



Problem 1: Prove that the length of median AM in triangle ABC is greater than (AB+ AC - BC)/2.

TRIANGLE INEQUALITY





$AB+AC > BC \quad |AB-AC| < BC$

$AC+BC > AB \quad |AC-BC| < AB$

$AB+BC > AC \quad |AB-BC| < AC$

$AM > AB-BM$

$+ \quad AM > AC-CM$

$2AM > AB+AC-BC$

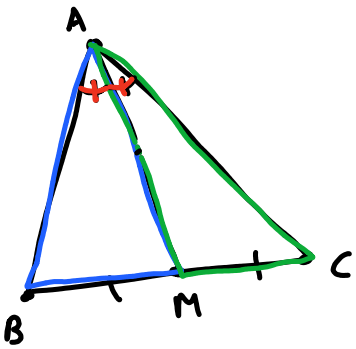
$BM = CM$

$-BM-CM$

$-(BM+CM)$

$$AM > \frac{AB+AC-BC}{2}$$

Problem 2: In triangle ABC the median AM is longer than half of BC. Prove that angle BAC is acute. $\angle BAC < 90^\circ$



$$AM > \frac{BC}{2}$$

$AM > BM$

$AM > CM$

In $\triangle ABM$:

$\angle BAM + \angle AMB + \angle MBA = 180^\circ$

In $\triangle ACM$:

$\angle AMC + \angle MCA + \angle CAM = 180^\circ$

In $\triangle ABC$:

$\angle ABC + \angle BCA + \angle CAB = 180^\circ$

$\angle BAM + \angle CAM = \angle CAB$

$$AM = \sqrt{\frac{2AB^2+2AC^2-BC^2}{4}}$$

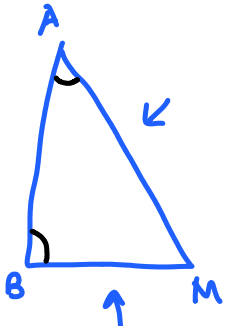
Sukaeb's solution

$BAM < ABC$

$CAM < ACB$

$BAC < ABC + ACB$

If ... $BAC > 90^\circ$:



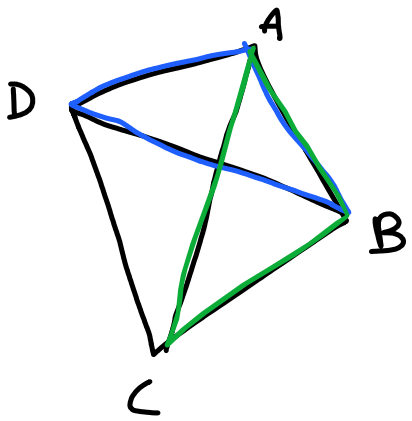
Assume

$\angle ABC + \angle ACB > 90^\circ$

$\angle ABC + \angle ACB + \angle BAC > 180$ ✗ not true

✗ contradiction

Problem 3: Prove that ABCD is a convex quadrilateral and AB+ BD < AC



$AD > AC - CD$

$AB + BD > AD$

$AC + CD > AD$

$2AD < AB + AC + BC + CD$

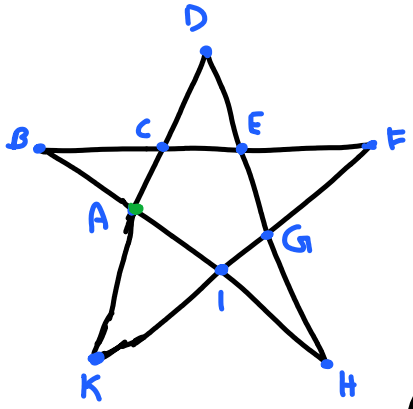
$AB + BD > AC - CD$

$AB + BD + CD > AC$

Homework

Problem 4: Is it possible for some five-pointed star ABCDEFGHIK to satisfy the inequalities: AB > BC, CD> DE, EF > FG, GH > HI, IK >KA?

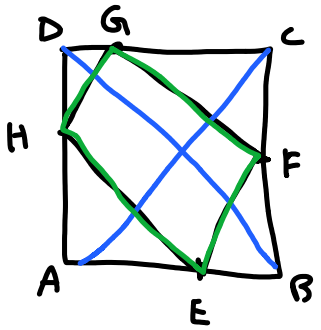
$AB + CD + EF + GH + IK < BC + DE + FG + HI + K$



Homework

ふA

Problem 5: A point is selected on each side of a square. Prove that the perimeter of the quadrilateral formed by these points is no less than twice the length of the square's diagonal.



perimeter of green quadrilateral

$HE < AH + AE$

$EF < BE + BF$

$GF < FC + GC$

$GH < DG + DH$

perimeter < perimeter

perimeter < diagonals × 2

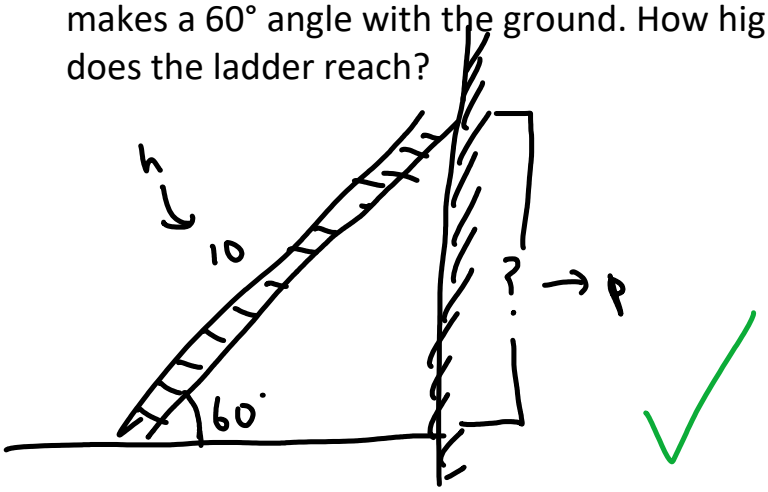
earlier proved:

2x diagonals > perimeter

Maths Circle Junior - Homework Problems

Saturday, May 31, 2025 9:31 AM

Problem 1: A ladder 10 feet long leans against a wall and makes a 60° angle with the ground. How high up the wall does the ladder reach?

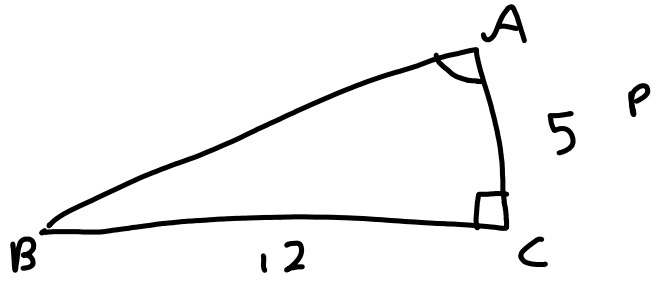


$$\sin 60 = \frac{p}{10}$$

$$p = \frac{\sqrt{3}}{2} \times 10$$

$$p = 5\sqrt{3}$$

Problem 2: In triangle ABC, angle C is a right angle, and BC = 12, AC = 5. Find the tangent of angle B.



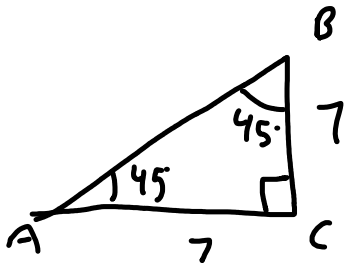
$\tan B = \frac{b}{5} = \frac{5}{12}$

sin = sine

cos = cosine

tan = tangent

Problem 3: In triangle ABC, angle C is a right angle, and angle A is 45°. If AC = 7, find the length of AB.



$7\sqrt{2}$

$\cos 45 = \frac{7}{h}$

$h = \frac{7}{\frac{1}{\sqrt{2}}}$

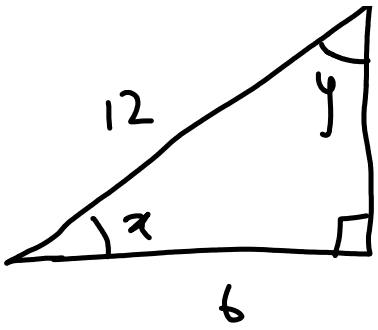
$= 7\sqrt{2}$

Problem 4: In a triangle, one side measures 6, the adjacent angle is [redacted], and the hypotenuse is 12. Find the sine of the angle opposite the side of length 6.

cosec = cosecant

sec = secant

cot = cotangent



$$\frac{\text{opp.}}{\text{hyp.}} = \frac{6}{12} = \frac{1}{2}$$

$\angle A$	0^o	
sin A	0	
cos A	1	
tan A	0	

Problem 5:
What is the least integer value of x such that
1 / (x + 2) > 1 / (x + 1)

30^o	45^o	60^o	90^o
$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

Problem 6:
If $a > b > 0$, prove that
$1 / a < 1 / b$

Problem 7:
Find all x that satisfy both:
$x^2 - 5x + 6 \geq 0$
and
$x - 1 < 2$

Maths Circle Junior - Homework Problems

Friday, February 28, 2025 7:13 PM

Problem 5: Prove that $2^n + 6 \cdot 9^n$ is always divisible by 7 for $\{n \in \mathbb{Z}^+\}$

n belongs to integers (positive)

Case 1: $n \equiv 0 \bmod 3 \rightarrow 2^n \equiv 1 \bmod 7$, $6 \cdot 9^n \equiv 6 \bmod 7 \rightarrow 2^n + 6 \cdot 9^n \equiv 7$

Case 2: $n \equiv 1 \bmod 3 \rightarrow 2^n \equiv 2 \bmod 7$, $6 \cdot 9^n \equiv 5 \bmod 7 \rightarrow 2^n + 6 \cdot 9^n \equiv 7$

Case 3: $n \equiv 2 \bmod 3 \rightarrow 2^n \equiv 4 \bmod 7$, $6 \cdot 9^n \equiv 3 \bmod 7 \rightarrow 2^n + 6 \cdot 9^n \equiv 7$

$2 \equiv 9 \bmod 7 \rightarrow 2^n \equiv 9^n \bmod 7 \rightarrow 2^n + 6 \cdot 9^n \equiv 7 \cdot 9^n \equiv 0 \cdot 9^n \equiv 0 \bmod 7$

Challenge Problem

Problem 6: Find x if $3^x - x^2$ is divisible by 5 for $\{x \in \mathbb{Z}^+, x \leq 100\}$.

number of values of

x belongs to integers (positive) where $x \leq 100$

$3^1 \equiv 3 \bmod 5$

$3^2 \equiv 4 \bmod 5$

$3^3 \equiv 2 \bmod 5$

$3^4 \equiv 1 \bmod 5$

$1^e \equiv 1 \bmod 5 \rightarrow 2 \bmod 5$

$2^2 \equiv 4 \bmod 5 \rightarrow 0 \bmod 5$

$3^2 \equiv 4 \bmod 5 \rightarrow -2 \bmod 5 \equiv 3 \bmod 5$

$4^2 \equiv 1 \bmod 5 \rightarrow 0 \bmod 5$


L paw
L paw
L paw

Miscellaneous Problem

Problem 7: If $a+b+c=0$, prove that $a^3+b^3+c^3=3abc$

Problem 7: Prove the AM-GM inequality.

Problem 7: Prove that if we decrease by 7 the sum of the squares of any three natural numbers, then the result cannot be divisible by 8.

Maths Circle Junior - Inequalities

Friday, January 17, 2025 9:54 PM

Problem 1. Which number is greater: 31^{11} or 17^{14}

$31^{11} < 32^{11}$

$32^{11} = (2^5)^{11} = 2^{55}$

$2^{55} < 2^{56}$

$2^{56} = (2^4)^{14} = 16^{14}$

$16^{14} < 17^{14}$

$\therefore 31^{11} < 17^{14}$

Problem 2. Which number is greater:

a) 2^{300} or 3^{200}

b) 2^{40} or 3^{28}

c) 5^{44} or 4^{53}

a) $2^{300} = (2^3)^{100} = 8^{100}$

$3^{200} = (3^2)^{100} = 9^{100}$

b) $2^3 < 3^2$

$2^{39} < 3^{26}$

$2^{40} < 3^{28}$

c) $5^4 = 625$ $4^5 > 5^4$

$4^5 = 1024$ $4^{40} > 5^{40}$

$4^5 > 5^4$ $4^{\frac{3}{2}} 4^{40} > 5^4 \times 5^{40}$

$4^{53} > 5^{44} \rightarrow (4^5)^{11} (5^4)^{11}$

$4^{53} > 5^{44}$ $(1024)^{11} > (625)^{11}$

Problem 3: Which number is greater: $1234567 \cdot 1234569$ or 1234568^2?

$x+1$ x $x-1$

$(x-1)(x+1) = x^2 - 1 < x^2$

1234568^2

Problem 4: Prove that $2^{100} + 3^{100} < 4^{100}$

$2^{100} < 3^{100}$

$3^{100} \times 2 < 4^{100}$

$3^{100} + 3^{100} < 4^{100}$

$2^{100} + 3^{100} < 4^{100}$

Problem 5: Prove that $4^{79} < 2^{100} + 3^{100} < 4^{80}$

Problem 6: How many digits does the number 2^{1000} have?

Problem 7: Prove the AM-GM inequality.

Maths Circle Junior - Miscellaneous

Saturday, August 23, 2025 9:34 AM

1) How many digits does $8^5 \times 5^{10} \times 15^5$ have?

$2^{15} \times 3^5 \times 5^{15} = 10^{15} \times 3^5$

$= 243 \times 10^{15}$

$= 2.43 \times 10^{17}$

243 . . . . . (18 digits)

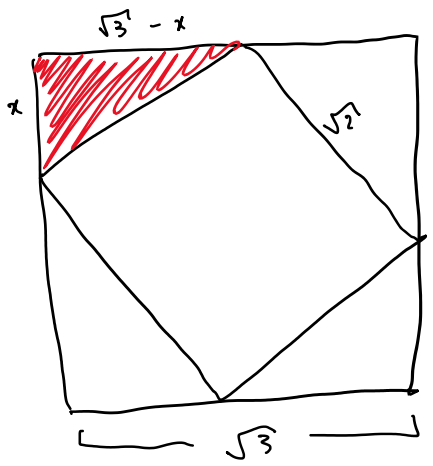
18 digits

2) How many positive perfect squares less than 2023 are divisible by 5?

$\lfloor \sqrt{2023} \rfloor = 44$

$\lfloor \frac{44}{5} \rfloor = \boxed{8}$

3) A square of area 2 is inscribed in a square with area 3. What is the ratio of the shorter leg to longer leg in the triangle?



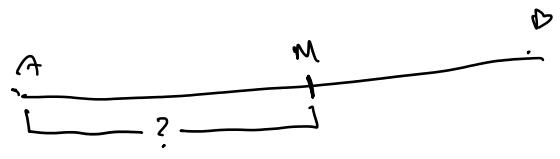
$x^2 + (\sqrt{3} - x)^2 = (\sqrt{2})^2$

$2x^2 - 2\sqrt{3}x + 1 = 0$

$x = \frac{\sqrt{3} \pm 1}{2}$

$\frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

4) Cities A and B are 45 km apart. Alicia lives in A and Beth lives in B. Alicia bikes towards B at 18 km/hr, and Beth bikes towards A at 12 km/hr. How many km from City A will they be when they meet?



27

5) $x \diamond y = |x - y|$

$(1 \diamond (2 \diamond 3)) - ((1 \diamond 2) \diamond 3) = ?$

$= |1 - |2 - 3|| - ||1 - 2| - 3|$

$= |1 - 1| - |1 - 3|$

$= 0 - 2$

$= -2$

Maths Circle Junior - Miscellaneous

Friday, April 11, 2025 11:28 PM

Problem 1: A positive integer leaves a remainder of 2 when divided by 5 and a remainder of 3 when divided by 7. What is the smallest such number?

7, 12, 17, 22, ...

10, 17, 24, ...

Problem 2: How many integers between 1 and 1000 are divisible by neither 2 nor 5?

500 → divisible by 2

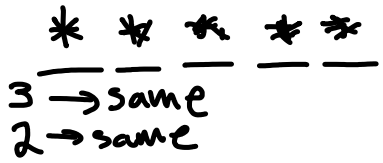
200 → divisible by 5

100 → divisible by 10

500 + 200 − 100 = 600 → divisible by 5 & 2

1000 − 600 = 400

Problem 3: In how many ways can the letters of the word SUCCESS be arranged?



$$\frac{5!}{2! \cdot 3!}$$

$$\frac{7 \times 6 \times 5 \times 4^{2}}{3! \times 2!}$$

$$= 42 \times 10$$

$$= 420$$

Problem 4: The sum of three consecutive even integers is 222. What is the product of the smallest and largest of these integers?

$$2x + 2x + 2 + 2x + 4 = 222$$

$$x = 36$$

$$2x = 72$$

$$2x + 4 = 76$$

$$72 \times 76 = 5472$$

Problem 5: A jar contains red, blue, and green marbles. There are 6 red marbles and twice as many blue as green marbles. If there are 30 marbles in total, how many blue marbles are there?

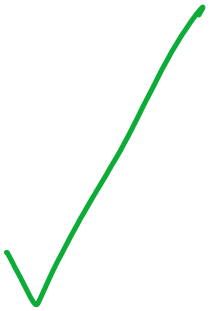
green → x
blue → $2x$

$2x + x + 6 = 30$

$3x = 24$

$x = 8$

$2x = 16$



Problem 6: Let x and y be real numbers such that $x + y = 5$ and $x^2 + y^2 = 13$. What is the value of xy?

$x + y = 5$

$x^2 + y^2 = (x+y)^2 - 2xy$

$13 = 5^2 - 2xy$

$2xy = 12$

$xy = 6$

Problem 7: Two people start at opposite ends of a 100-meter track and run toward each other. One runs at 5 m/s and the

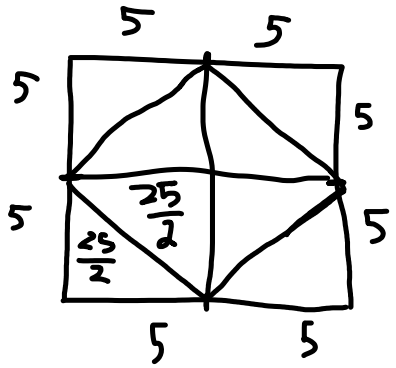
other at 3 m/s. How many seconds until they meet?

1 second → 5+3=8

$8ms^{-1}$

160m → $\frac{160}{8}$ = 12.5 ✓

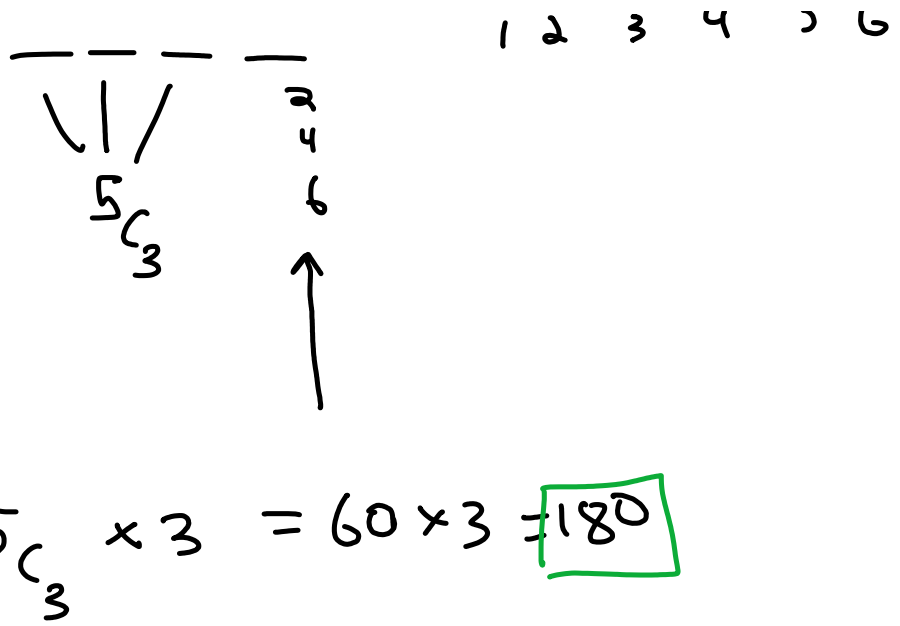
Problem 8: A square has side length 10. A second square is formed by connecting the midpoints of the sides of the original square. What is the area of the smaller square?



$a^2+b^2=c^2$

$\frac{25}{2} \times 4 = 50$ ✓

Problem 9: How many different 4-digit numbers can be formed using the digits 1 through 6 if repetition is not allowed and the number must be even?



$_5C_3 \times 3 = 60 \times 3 = 180$

Problem 10: How many distinct values are there in the set of all possible products ab where a and b are integers from 1 to 10 inclusive?

Maths Circle Junior - Miscellaneous

Friday, June 20, 2025 11:40 PM

Problem 1:
The sum of a number (x) and its reciprocal is 5. What is the minimum possible value of the sum of the number and its cube?

$$x + \frac{1}{x} = 5$$

$$x^2 + 1 = 5x$$

$$x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 4}}{2}$$

✓

$$= \frac{5 \pm \sqrt{21}}{2}$$

$$x + x^3 = \boxed{0.21}$$

Problem 2:
The digits of a three-digit number are in arithmetic progression. If the number is divisible by 9, what is the sum of all such numbers?

369

135

5328

234

5 7

4 8

Problem 3:
What is the smallest positive integer n such that n leaves a remainder of 3 when divided by 4, a remainder of 4 when divided by 5, and a remainder of 5 when divided by 6?

LCM(3,4,5) = 60

60 - 1 = 59

Problem 4:
A square and a circle have the same area. The side length of the square is doubled. By what percent does the area of the new square exceed the area of the original circle?

$x^2 \leftarrow$

$4x^2 \leftarrow$

$$\left(\frac{4x^2 - x^2}{x^2}\right) \times 100\%$$

$$= 300\%$$

Problem 5:
A function f satisfies f(x + 1) = 2 * f(x) and f(0) = 3. What is the value of f(5)?

Problem 6:
How many different four-letter "words" can be formed using the letters A, B, C, and D if each letter can be used more than once and the word must contain at least one B?

Problem 7:
Find all positive integers x such that x squared plus 19 times x plus 48 is a perfect square.

Problem 8:
The sum of the squares of three consecutive integers is 365. What is the middle integer?

Problem 9:
How many three-digit integers between 100 and 999 have all digits different and in increasing order?

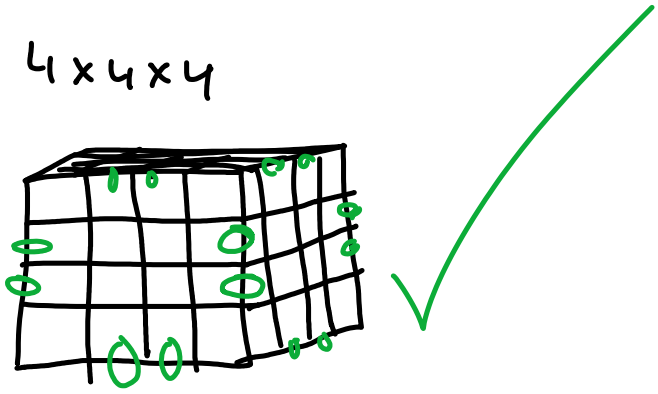
Problem 10:
A set of five consecutive positive integers has a product divisible by 120. What is the smallest possible value of the smallest number in the set?

Maths Circle Junior - Miscellaneous

Friday, April 4, 2025 11:39 PM

Problem 1: A cube has all of its faces painted. It is then cut into 64 smaller, equal-sized cubes. How many of the small cubes have exactly two painted faces?

$4 \times 4 \times 4$



Problem 2: Alan writes down five consecutive positive integers. The sum of the squares of the first x and last numbers is 458. What is the middle number?

$x+4$

Let the smallest number be x

$$x^2 + (x+4)^2 = 458$$

$$x^2 + x^2 + 8x + 16 = 458$$

$$2x^2 + 8x - 442 = 0$$

$x^2 + 4x - 221 = 0$

$x = 13$ or -17

middle $= x+2 = 13+2 = \boxed{15}$

Problem 3: A rectangular piece of paper measures 8 inches by 15 inches. It is rolled up so that the two shorter sides meet to form a cylinder. What is the volume of the cylinder?

8 inch → 8 inch

circumference = 15

radius $= \frac{15}{2\pi} = \frac{7.5}{\pi}$

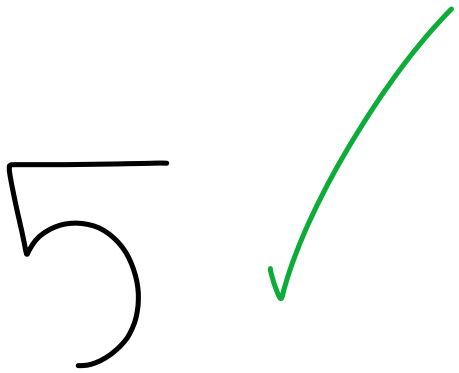
volume $= \pi r^2 h = \pi\left(\frac{7.5}{\pi}\right)^2 (8) = \frac{450}{\pi}$

Problem 4: How many integers between 100 and 999 have digits that sum to 24?

24 → (6,9,9) (7,8,9) (8,8,8)

3 + 6 + 1 = 10

Problem 5: A pizza was cut into 12 equal slices. Jenny ate one-third of them, Carlos ate one-fourth of them, and Keisha ate the rest. How many slices did Keisha eat?



Problem 6: There is a fair six-sided die which is rolled 3 times. What is the probability that all numbers rolled are different?

$$\frac{120}{216} = \frac{5}{9}$$

Problem 7: How many rectangles of any size can be formed using only the gridlines of a 5x5 grid?

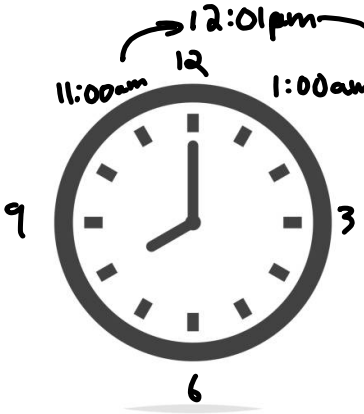
$${}^{6}C_{2} \times {}^{6}C_{2} = 15 \times 15 = \boxed{225}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \rightarrow ax^2+bx+c$$

Maths Circle Junior - Modular Arithmetic

Sunday, February 9, 2025 9:31 AM

What is the meaning of mod?



$13 \equiv 1 \mod 12$

$16 \equiv 4 \mod 12$

a b n

$a-b = 16-4 = 12$, $n = 12$

$a = 16$

$b = 12$

$12\overline{)16}\,(1$

-12

4 → b

$a \equiv b \mod n$

↳ $a-b$ is divisible by n

↳ b is the remainder when a is divided by n

Modular Laws

- $a \equiv a \mod m$ (reflectivity)
- if $a \equiv b \mod m$, then $b \equiv a \mod m$ (symmetry)
- if $a \equiv b \mod m$ and $b \equiv c \mod m$, then $a \equiv c \mod m$ (transitivity)

Problem 1: Find the tens digit of 7^{2011}.

$7^1 \equiv 07 \mod 100$

$7^2 \equiv 49 \mod 100$

$7^3 \equiv 43 \mod 100$

$7^4 \equiv 01 \mod 100$

$\vdots$

$7^{2011} = (7^4)^{502} \times 7^3 \equiv 7^3 = 343$

$\downarrow$

$\boxed{4}$

Problem 2: Given that $5x \equiv 6 \mod 8$, find x

where $\{x \in \mathbb{Z}^+\}$

x belongs to integers (positive)

$5 \times 1 \equiv 5 \mod 8$

$5 \times 2 = 10 \equiv 2 \mod 8$

$5 \times 3 = 15 \equiv 7 \mod 8$

$5 \times 4 = 20 \equiv 4 \mod 8$

$5 \times 5 = 25 \equiv 1 \mod 8$

$5 \times 6 = 30 \equiv 6 \mod 8$

$5x \equiv 6 \mod 8$

$25x \equiv 30 \mod 8$

$x \equiv 30 \mod 8$

$8)\overline{30}(3$

-24

$\boxed{6}$

Problem 3: In year N, the 300th day of the year is a Tuesday. In year N+1, the 200th day is also a Tuesday. On what day of the week did the 100th day of the year N-1 occur?

$65+200=265$, $66+200=266$ days

in between 300th of N and 200th of N+1

$266 \equiv 0 \bmod 7$ → N+1 is leap year, N-1 is not

$265+300=565$ → 300th of N and 100th of N-1

$565 \equiv 5 \bmod 7$ → Tuesday

M

S

S

F

Thursday

Problem 4: Find $\sum$ number of values of n if $\{n \in \mathbb{Z}^+, 1 \le n \le 25\}$ such that n^2+3n+2 is divisible by 6.

↳ n belongs to integers (positive), where n is between 1 and 25 inclusive

$n^2+3n+2 \equiv 0 \bmod 6$

$(n+1)(n+2) \equiv 0 \bmod 6$

$2 \times 3 \equiv 5 \times 6 \equiv 6 \times 1 \bmod 6$

$n+1 \equiv 2, 5, 6 \bmod 6$, $n \equiv 1, 4, 5 \bmod 6$ → $5+4+4=13$ numbers

n ... $2^n+6\cdot9^n$ is always divisible by 7

Problem 5: Prove that [illegible]
for $\{n \in \mathbb{Z}^+\}$

Challenge Problem

Problem 6: Find x if $3^x - x^2$ is divisible by 5 for $\{x \in \mathbb{Z}^+, x \leq 100\}$.

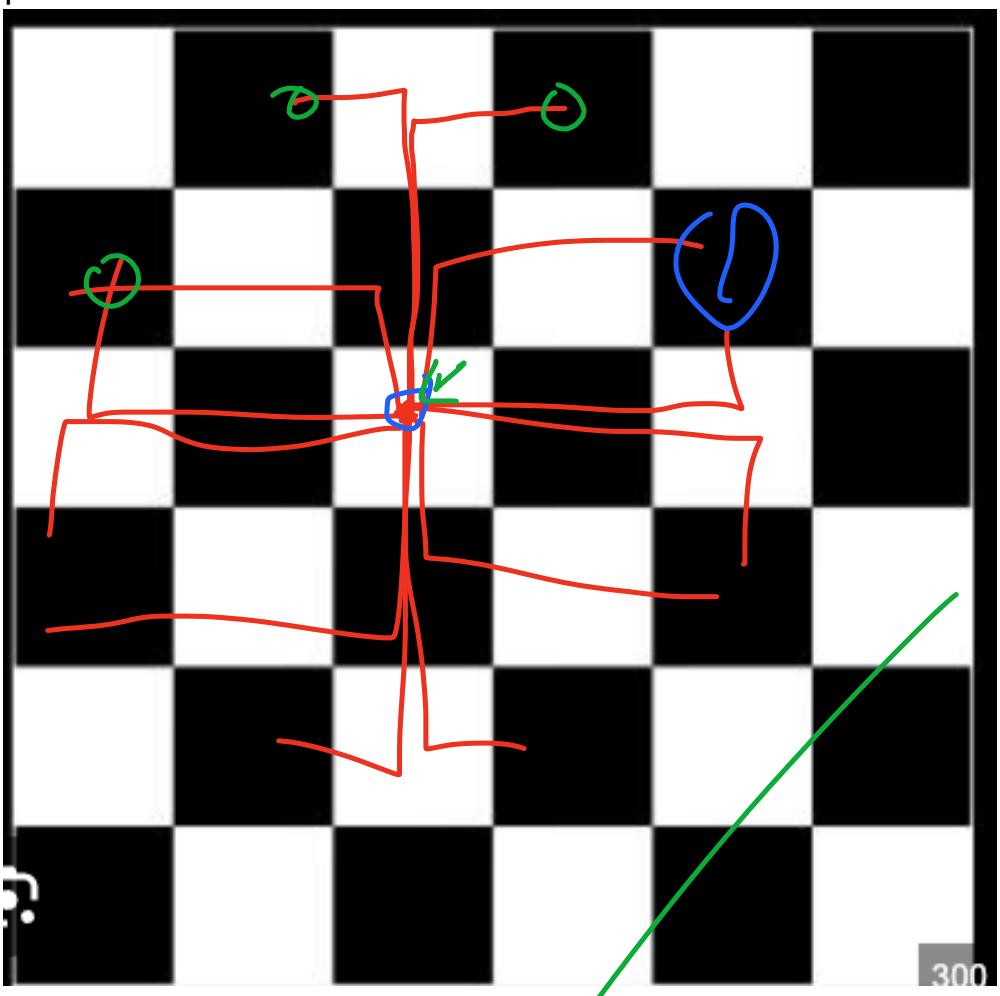
Miscellaneous Problem

Problem 7: If $a+b+c=0$, prove that $a^3+b^3+c^3 = 3abc$

Maths Circle Junior - Parity

Friday, May 2, 2025 10:45 PM

Problem 1: A 6×6 grid is colored in a checkerboard pattern, starting with black in the top-left corner. A knight is placed on a black square. Is it possible for the knight to visit every square of the grid exactly once and return to its starting point?



knight is always moving to the opposite

Problem 2: A pile of 15 stones is on the table. Two players take turns removing 1, 2, or 3 stones. The player who removes the last stone wins. If both play optimally, who wins: the first or second player?

$15 \equiv 3 \bmod 4$

First player → 3

$12 \equiv 0 \bmod 4$

Problem 3: A number has digits that sum to 39. Can the number be divisible by 9 and still be even?

digits sum $\equiv 0 \bmod 9$

not divisible

Problem 4: You're given a 5×5 board with one square removed. You are given 12 L-shaped tiles, each covering 3 squares. Can you tile the rest of the board?

25 total

Problem 5: You repeatedly subtract the sum of the digits of a number from itself (e.g., 86 → 86 − (8 + 6) = 72 → 72 − (7 + 2) = 63, etc.). Will the parity of the result ever change?

Yes ✓

Problem 6: Let n be an integer such that $n^2 + n$ is divisible by 2. What can you say about the parity of n?

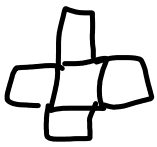
✓

Problem 7: A number n leaves remainder 1 when divided by 2, 2 when divided by 3, and 4 when divided by 5. What is the parity of n?

Problem 8: A number is written as the product of two consecutive integers. Is the product always even, always

odd, or does it depend?

Problem 9: Alice and Bob play a game on a 4×4 board where each move flips a chosen square and its adjacent squares (up/down/left/right). Starting from all white squares, is it always possible to make the board all black?

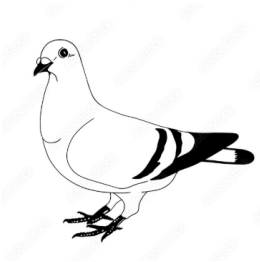


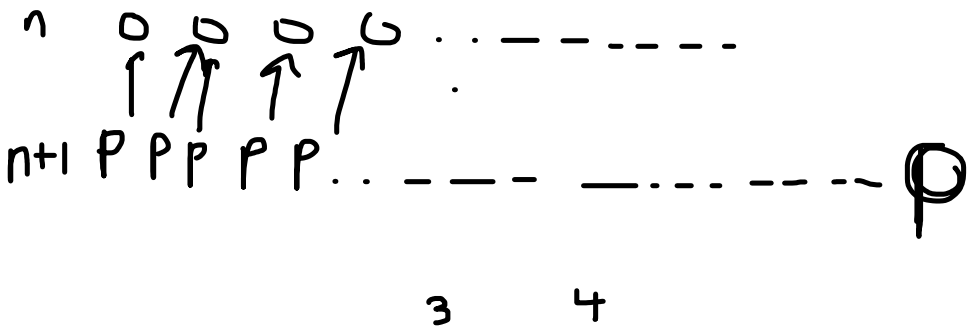
Problem 10: A positive integer is written as a sum of consecutive integers. If the number of terms is even, what can you say about the parity of the total sum?

Maths Circle Junior - PHP (Nov 30)

Saturday, November 23, 2024 7:20 PM

What is Pigeon Hole Principle?





Say we have N number of holes, and I have N + 1 pigeons: after putting all the pigeons Into a hole, there must be one hole that has more than 1 pigeon.

Example 1: A bag contains 10 black beads and 10 white beads. What is the smallest number of which must be drawn from the bag, without looking, so that among these beads, there are 3 pa opposite color.

beads
airs of

Example 2: One million pine trees grow in a forest. It is known that no pine tree has more than 6
pine needles on it. Show that two pine trees in the forest must have the same number of pine n

Write your solution using PHP:

Number of possible needles on a pine tree: holes
Number of pine trees: pigeons

Problem 3: Given twelve consecutive integers, show that two of them can be chosen whose diffe
divisible by 11.

Problem 4: For the 2026 FIFA World Cup, there will be M football teams, each of which has 11 play
All the players will be gathered at an airport for a trip to another country for an important game, b
they will be traveling on "standby". There are 10 flights to their destination, and it turns out that e
flight has room for exactly M players. One football player will take his own helicopter to the game
rather than traveling standby on a plane. Show that at least one whole team will be sure to get to
important game.

600000
needles.

s

ρ_1

$\rho_0 \rho_0$

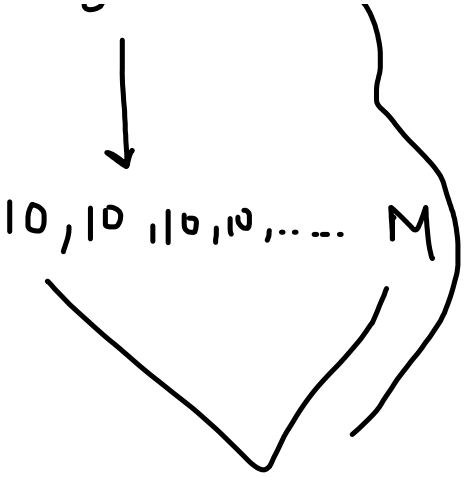
erence is

$= D$

N

$N+1$

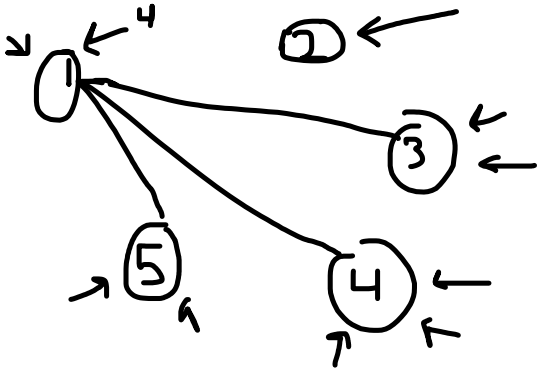
yers.
but
each
,
the



10M

holes

Problem 5: Show that in any group of five people, there are two who have an identical number o
friends within the group.



Case 1: 1, 2, 3, 4 → holes

Case 2: 0, 1, 2, 3 → holes

Case 1:

0000

P P P P P

Case 2:

0060

PPPPP

Homework:

PHP Problem 1: Ten students solved a total of 35 problems in a math circle. Each problem was solved by exactly one student. There is at least one student who solved exactly one problem, at least one student who solved exactly two problems, and at least one student who solved exactly three problems. Prove that there is also at least one student who has solved five problems.

PHP Problem 2: Show that an equilateral triangle cannot be covered completely by two smaller equilateral triangles.

PHP Problem 3: What is the largest number of kings which can be placed on chessboard so that no two of them put each other in check?

of

Miscellaneous Problem 1: The son of a professor's father is talking to the father of the professor's son, and the professor does not take part in the conversation. Is this possible?

Miscellaneous Problem 2: In a 3x3 square, fill all the boxes with the digits 1, 2, 3, 4,5, 6, 7, 8, and 9 such that all the sums of the numbers along the rows, columns, and two diagonals are equal.

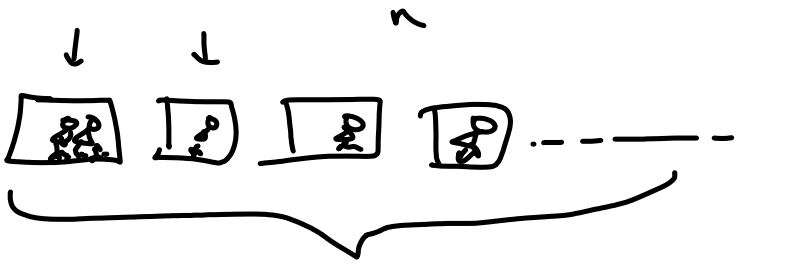
Miscellaneous Problem 3: In a certain year, there were exactly four Fridays and four Mondays in January. On what day of the week did the 20th of January fall that year?

Miscellaneous Problem 4: Cross out 10 digits from the number 123451234512345123451234512345 so that the remaining number is as large as possible.

Maths Circle Junior - PHP

Saturday, June 14, 2025 12:14 AM

Pigeon hole principle



at least 1 hole : 2 pigeons (at least)

how many pigeons do I need:

n+1 pigeons

Problem 1:
What is the minimum number of people required in a room to guarantee that at least two of them were born in the same month?

12 people → all different birthmonths

13

12+1=13

Problem 2:

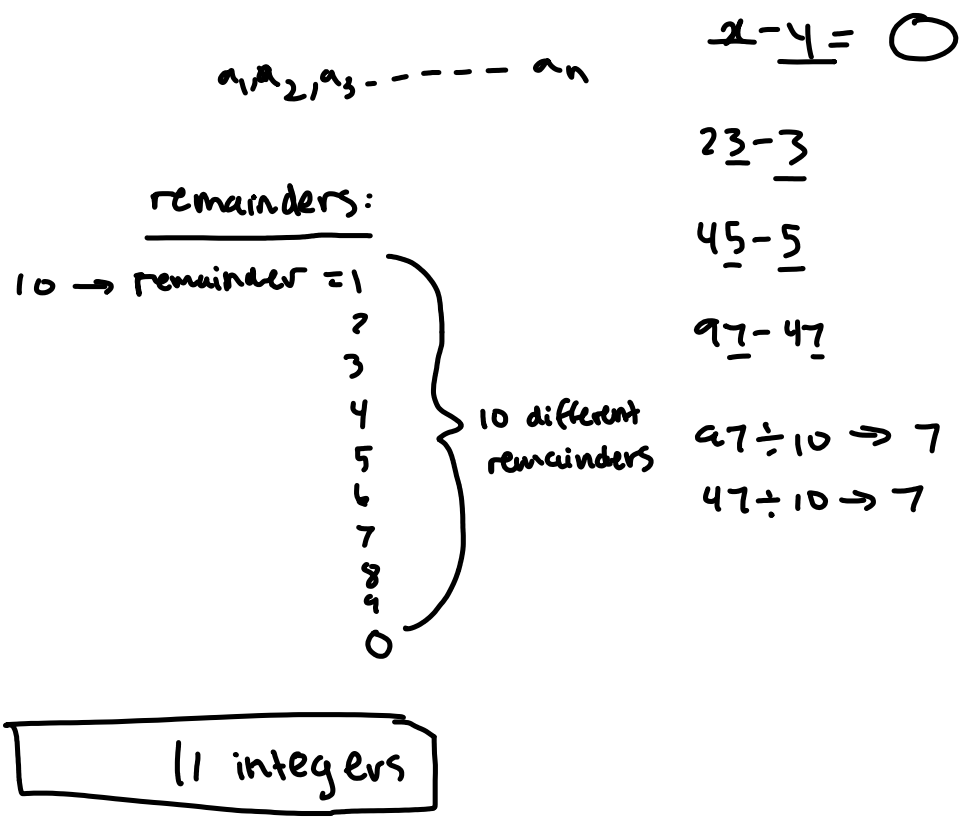
A drawer contains 10 red socks, 8 blue socks, and 6 green socks. What is the minimum number of socks one must draw to guarantee getting at least two of the same color?

Red
Blue
Green
R/B/G

Problem 3:

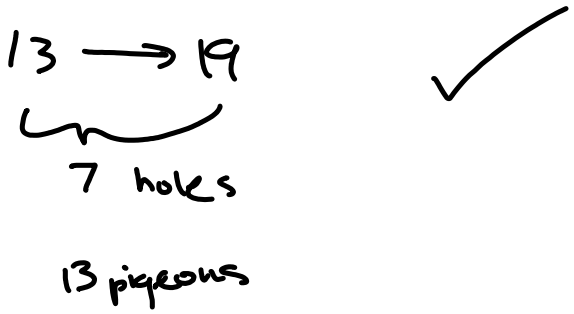
How many integers must be selected from the set {1, 2, 3, ...,

100} to guarantee that at least two of them differ by a multiple of 10?



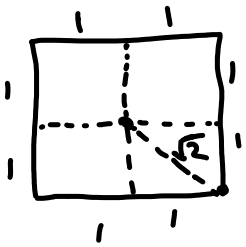
11 integers

Problem 4:
In a group of 20 students, each student shakes hands with at least 13 other students. Prove that at least two students shook hands with the same number of people.

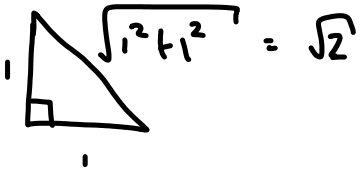


Problem 5:

Prove that in a 2x2 square if we place arbitrary 5 points, the maximum distance possible is $\sqrt{2}$ between adjacent



done



Problem 6:

A box contains n balls, each colored either red, green, blue, or yellow. What is the minimum number of balls you must choose to ensure at least 11 balls of one color?

41

Problem 7:
You are given 11 points in a 10 × 10 square grid. Prove that at least two of the points lie in the same 3 × 3 square.

Problem 8:
From the numbers 1 to 100 inclusive, how many must you choose to guarantee that two of them are consecutive integers?

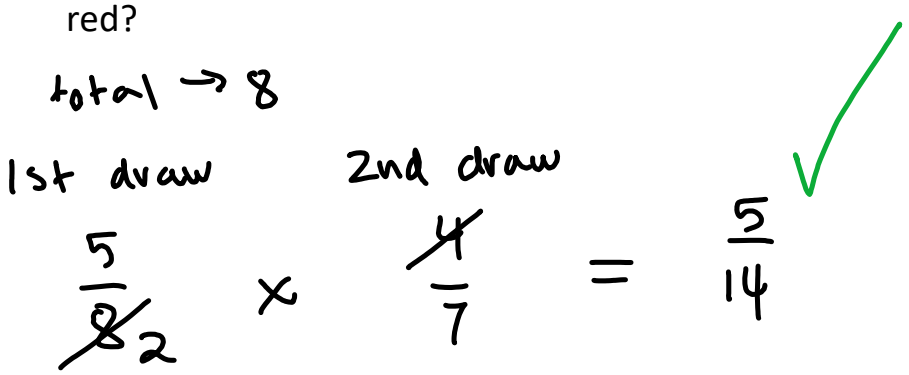
Problem 9:
You randomly select 21 integers from the set {1, 2, 3, ..., 40}. Prove that there exist two selected integers such that one divides the other.

Problem 10:
If you write down 51 distinct integers between 1 and 100, show that there must be at least two numbers that differ by 1.

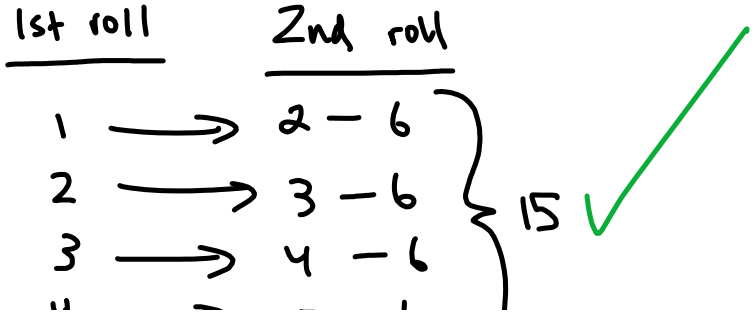
Maths Circle Junior - Probability

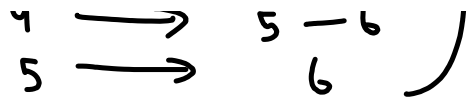
Friday, April 18, 2025 10:29 PM

Problem 1: A jar contains 5 red balls and 3 blue balls. One ball is drawn at random, not replaced, and then a second ball is drawn. What is the probability that both balls are red?



Problem 2: A standard 6-sided die is rolled twice. What is the probability that the second roll is greater than the first?





Problem 3: Three fair coins are flipped. What is the probability of getting exactly two heads?

H H T

H T H

T H H

$2^3 = 8$

$\frac{3}{8}$

Problem 4: A number is randomly chosen from the integers 1 through 100 inclusive. What is the probability that it is divisible by 3 or 5?

33 multiples of 3

20 multiples of 5

6 multiples of both

$$\frac{33+20-6}{100} = \frac{47}{100}$$

Problem 5: Two digits are chosen at random (without replacement) from the digits 1 through 9. What is the probability that their sum is odd?

even + odd = odd
odd + even = odd

1,3,5,7,9 — 5

2,4,6,8 — 4

$5 \times 4 \times 2 = 40$

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

$${}_9C_2 = \frac{9! \; 9 \times 8}{2! \; 7!} = 36$$

$$\frac{40}{36} = \frac{5}{9}$$

Problem 6: In a class of 12 students, 5 are wearing red shirts. If 3 students are chosen at random, what is the probability that at least one is wearing a red shirt?

12 − 5 = 7 not wearing shirt

$$\frac{7}{12} \times \frac{6}{11} \times \frac{5}{10} = \frac{7}{44}$$

$$1 - \frac{7}{44} = \frac{37}{44}$$

Problem 7: A bag contains 4 white, 3 black, and 2 green marbles. Two marbles are drawn at random without replacement. What is the probability that both are the same color?

$${}_9C_2 = 36 \quad (\text{total})$$

$${}_4C_2 + {}_3C_2 + {}_2C_2 = 6+3+1 = 10$$

$$\frac{10}{36} = \frac{5}{18}$$

Problem 8: A 3-digit number is formed by randomly selecting digits from 1 through 9 (no 0). What is the probability that all three digits are different?

same e

$$\_\_ \times \_\_ \times \_\_ = 729$$

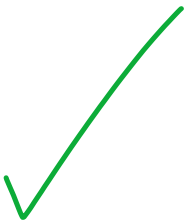
$9 \times 8 \times 7 = 504$ ✓

$$\frac{504}{729} = \frac{56}{81}$$

Problem 9: A spinner is divided into 5 equal sectors numbered 1 through 5. If you spin it twice, what is the probability that the product of the two spins is even?

odd

1, 3, 5 → 3 options



$$\frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$$

$$1 - \frac{9}{25} = \frac{16}{25}$$

Problem 10: A fair 4-sided die is rolled three times. What is the probability that all three results are different?

Maths Circle Junior - Remainders in Algebra

Friday, January 10, 2025 9:13 PM

Retry: Prove that the number $n^3 + 2n$ is divisible by 3 for any natural number n.

Case 1: $n \to 0$

$n^3 \to 0$, $2n \to 0$, $n^3 + 2n \to 0$

Case 2: $n \to 1$

$n^3 \to 1$, $2n \to 2$, $n^3 + 2n \to 1 + 2 = 3 = 0$

Case 3: $n \to 2$

$n^3 \to 2$, $2n \to 4 = 1$, $n^3 + 2n \to 2 + 1 = 3 = 0$

Problem 1: Prove that the number $n^5 + 4n$ is divisible by 5 for any natural number n.

Case 1: 0, $n^5 \to 0$, $4n \to 0$, $n^5 + 4n \to 0$

Case 2: 1, " $\to 1$, " $\to 4$, " " $\to 1 + 4 = 5 = 0$

Case 3: 2, " $\to 2$, " $\to 8 = 3$," " $\to 2 + 3 = 5 = 0$

Case 4: 3, " $\to 3$, " $\to 12 = 2$, " " $\to 3 + 2 = 5 = 0$

Case 5: 4, " $\to 4$," $\to 16 = 1$, " " $\to 4 + 1 = 5 = 0$

Problem 2: Prove that the number $n^2 + 1$ is not

Problem 2: Prove that the number $n^3 + 1$ is not divisible by 3 for any natural number n.

$n^2 \div 3 \rightarrow R = 0, 1$

$(n^3+1) \div 3 \rightarrow R = \boxed{1, 2}$

$3 \equiv 2 \bmod x$

$x = 1 \quad x = -1$

$3 = (2 \times 1) + 1$

$3 = (2 \times 2) - 1$

$a \equiv b \bmod c$

$c \div a - b \rightarrow 0$

Problem 3: Prove that the number $n^3 + 2$ is not divisible by 9 for any natural number n.

$n \equiv 0 \pmod 9 \longrightarrow n^3 \equiv 0^3 \bmod 9 \rightarrow 0 + 2 = 2$

$n \equiv 1 \pmod 9 \longrightarrow n^3 \equiv 1 \bmod 9 \rightarrow 1 + 2 = 3$

$n \equiv 2 \pmod 9 \longrightarrow n^3 \equiv 2^3 \bmod 9 \rightarrow 8 + 2 = 10 \rightarrow 1$

$n \equiv 3 \pmod 9 \longrightarrow n^3 \equiv 0 \bmod 9 \rightarrow 0$

1

8

0

1

8

$n \equiv 8 \pmod 9 \longrightarrow$

Problem 4: Prove that the number $n^3 - n$ is divisible by 24 for any odd n.

$n^3 - n = n(n^2 - 1) = n(n+1)(n-1)$

$n-1, n, n+1$

$24 = 2^3 \times 3$

$8 \times 3 = \boxed{24}$

odd number = $2k+1$

$2k+2$ ↓ even

$2k$ ↓ even

divisibility by 4

$2k = 2(k)$

$2k+2 = 2(k+1)$

Problem 5: Prove that the number $n^5 + 4n$ is divisible by 5 for any natural number n

Problem 6: Three prime numbers p, q, and r, all greater than 3, form an arithmetic progression: p = p, q = p + d, and r = p + 2d. Prove that d is divisible by 6.

Problem 7: Prove that if we decrease by 7 the sum of the squares of any three natural numbers, then the result cannot be divisible by 8.

Maths Circle Junior - Speed and Distance

Saturday, July 26, 2025 9:32 AM

Problem 1:
Alice runs at a constant speed of 6 km/h. Bob walks at 4 km/h and starts 15 minutes before Alice. How many minutes after Alice starts will she catch up to Bob?

Bob: 60 mins → 4km
15 mins → 1km

6-4 = 2 km/hr

every hour, Alice will be 2km closer to Bob

2km → 60mins
1km → 60mins / 2 = 30 mins

Problem 2:
A train travels 60 km at a speed of 40 km/h and then returns over the same distance at a speed of 60 km/h. What is the train's average speed over the entire trip?

40km → 1hr
1 → 1/40
60 → 1/40 × 60 = 3/2 = 1.5

60km → 1hr
1.5+1 = 2.5 hrs (total hours)

average speed = 120/2.5 = 48 km/h

Problem 3:
A cyclist travels from town A to town B, a distance of 48 km. On the way to B, she travels at 12 km/h. On the return trip, she travels at 16 km/h. What is her average speed for the entire trip?

total distance = 2×48km
total time = 48/12 + 48/16 = 7
average speed = 96/7 ≈ 13.71 km/hr

Problem 4:
Two cars start from opposite ends of a 300 km road and travel towards each other. One car travels at 60 km/h, the other at 40 km/h. How many minutes will it take before they meet?

A 60 → ← 40 B
300 km

60+40 = 100 km/hr
100 km → 1hr
300 km → 3hr

Problem 5:
A car travels for 2 hours at 60 km/h, then for 3 hours at 80 km/h. What is the car's average speed for the entire trip?

Problem 6:
A bus travels from City A to City B, a distance of 150 km. After traveling for 2 hours, it takes a 30-minute break, then continues at a speed 20 km/h faster than before. If the total trip takes 4 hours including the break, what was the speed before the break?

initial speed = x km/hr

First part: distance = 2x
speed after break: x+20
time = (150-2x)/(x+20)

(150-2x)/(x+20) + 2 + 1/2 = 4
x = 140/7

Problem 7:
Two people start biking toward each other from towns that are 120 km apart. One travels at 25 km/h, and the other at 15 km/h. After 1 hour, a bird starts flying from one biker to the other at 40 km/h, turning around each time it reaches one. How far has the bird flown when the bikers meet?

Problem 8:
A runner completes a 10 km race by running the first 6 km at 12 km/h and the last 4 km at 8 km/h. What is his average speed for the whole race?

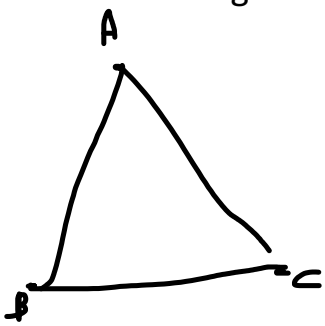
Problem 9:
Train A leaves a station traveling east at 72 km/h. Two hours later, Train B leaves the same station traveling east at 90 km/h. After how many hours from Train B's departure will it catch up to Train A?

Problem 10:
A man drives 100 km to a city. On the way there, he averages 50 km/h due to traffic. On the way back, he averages 100 km/h. What is his average speed for the entire round trip?

Maths Circle Junior - Triangle Inequality (Dec 7)

Saturday, December 7, 2024 8:13 PM

What does the triangle inequality say?



$AB < AC + BC$

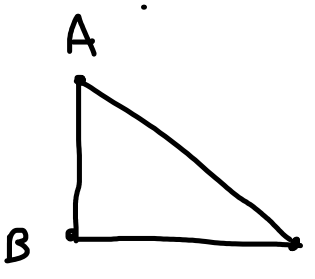
$AC < AB + BC$

$BC < AB + AC$

$AB > |AC - BC|$

$|-3| = 3$

Problem 1. Prove that for any three points A, B, and C we have $AC \geq |AB-BC|$.



$AB + BC \geq AC$

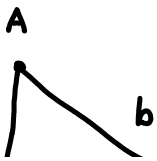
$BC + AC \geq AB$ ✓

$AB + AC \geq BC$ ✓

$AC \geq |BC - AB|$ ← equal

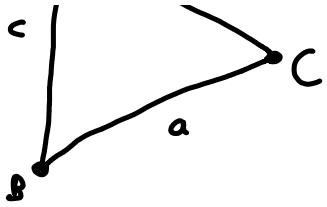
$AC \geq |AB - BC|$ ↩

Problem 2: Prove that the length of any side of a triangle is not more than half its perimeter.



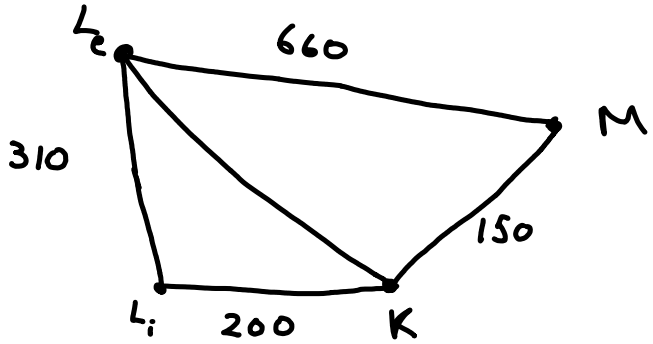
$a < b + c$

$2a < a + b + c$



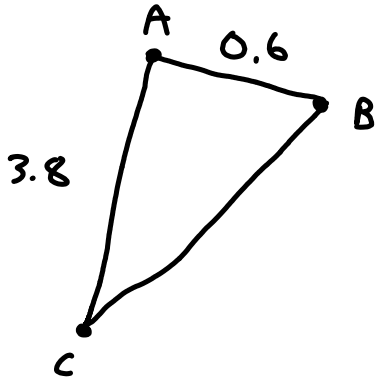
$a < \frac{P}{2}$ ✓

Problem 3: The distance from Leningrad to Moscow is 660 kilometers. From Leningrad to the to
Likovo it is 310 kilometers, from Likovo to Klin it is 200 kilometers, and from Klin to Moscow is 1
kilometers. How far is it from Likovo to Moscow?



Problem 4: Side AC of triangle ABC has length 3.8, and side AB has length 0.6. If the length of side BC is an integer, what is this length?

$3.2 < BC < 4.4$

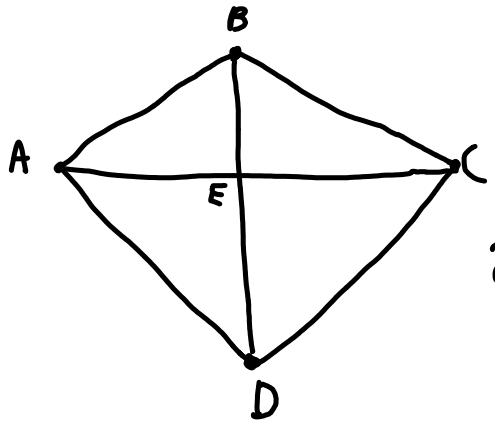


$BC = 4$

own of
150

Problem 5: Prove that the sum of the diagonals of a convex quadrilateral is less than the perimeter but more than half the perimeter.

Convex quadrilateral: all angles in the quadrilateral < 180, sum of angles = 360



$AC < AB + BC$

$AC < AD + CD$

$2AC < AB + BC + AD + CD$

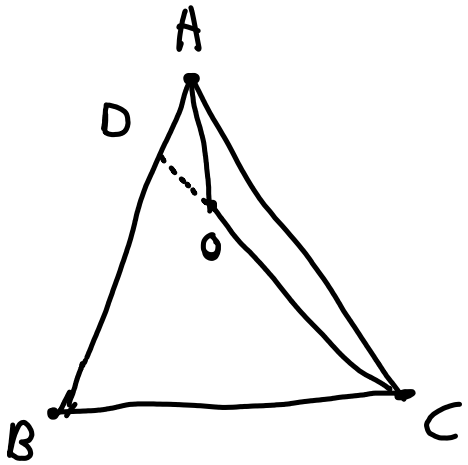
$AC < \frac{P}{2}$

$\frac{P}{2} < AC + BD <$

from same approach

$BD < \frac{P}{2}$

Problem 6: If point O is inside a triangle ABC, prove that AO + OC < AB + BC.



D, O, C are collinear

$\triangle AOC, \triangle ADC$

Homework


P

Homework:

Triangle Inequality Problem 1: Prove that the sum of the diagonals of a convex pentagon is greater than the perimeter but less than double the perimeter.

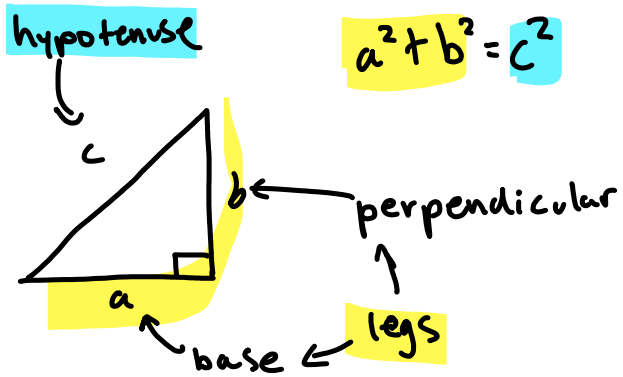
Triangle Inequality Problem 2: Prove that the distance between any two points inside a triangle is not greater than half the perimeter of the triangle.

Triangle Inequality Problem 3: Find a point inside a convex quadrilateral such that the sum of the distances from the point to the vertices is minimal.

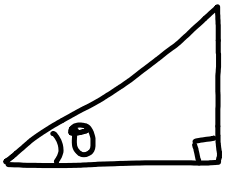
Maths Circle Junior - Trigonometry

Friday, May 9, 2025 10:40 PM

Pythagorean Theorem:



Trigonometric functions:



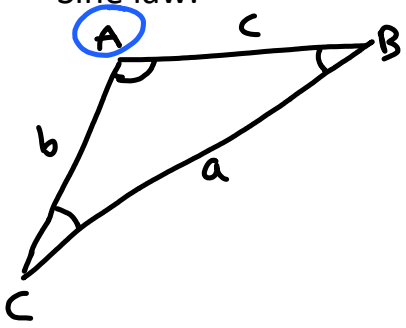
$\sin\theta = \frac{p}{h}$

$\cos\theta = \frac{b}{h}$

$\tan\theta = \frac{p}{b}$

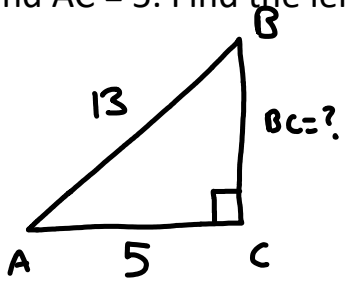
$\angle A$	0^o	30^o	45^o	60^o	90^o
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

Sine law:



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

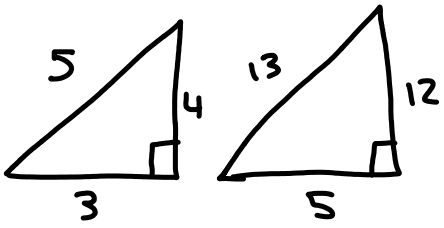
Problem 1: In triangle ABC, angle C is a right angle, AB = 13, and AC = 5. Find the length of BC.



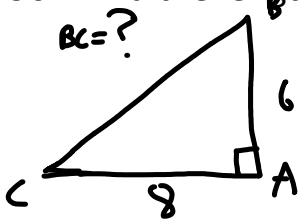
$BC^2 + 5^2 = 13^2$

$BC^2 = 169 - 25 = 144$

$BC = 12$



Problem 2: Triangle ABC has AB = 6, AC = 8, and angle A = 90°. Find the length of BC.



1st way

$BC^2 = 6^2 + 8^2$

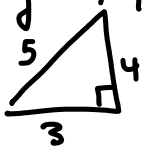
$= 36 + 64$

$= 100$

$BC = 10$

2nd way

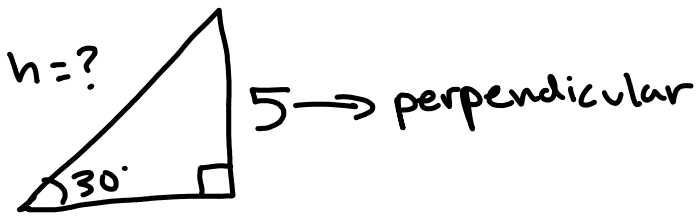
using 3,4,5 triangle



6 and 8 is 3,4,5 triangle multiplied by 2

$\therefore, BC = 5 \times 2 = 10$

Problem 3: In a triangle, one angle measures 30°, and the side opposite it has length 5. Another angle measures 90°. Find the length of the hypotenuse.

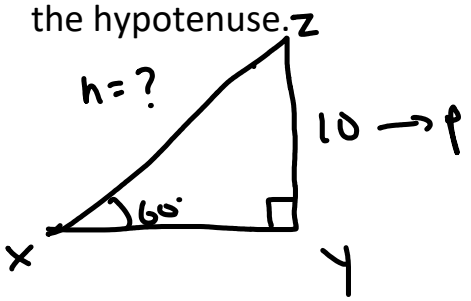


$$\sin\theta = \frac{P}{h}$$

$$\sin 30 = \frac{5}{h}$$

$$h = \frac{5}{\sin 30} = \frac{5}{\frac{1}{2}} = 10$$

Problem 4: In right triangle XYZ with right angle at Y, angle X = 60°, and side opposite angle X is 10. Find the length of the hypotenuse.



$$\sin\theta = \frac{Y}{h}$$

$$\sin 60 = \frac{10}{h}$$

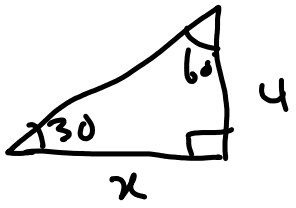
$$h = \frac{10}{\frac{\sqrt{3}}{2}}$$

$$= 10 \times \frac{2}{\sqrt{3}}$$

$$= \boxed{\frac{20}{\sqrt{3}}}$$

Problem 5: A triangle has angles 30°, 60°, and 90°, and the

Problem 5: A triangle has angles 30°, 60°, and 90°, and the length of the side opposite the 30° angle is 4. Find the length of the side opposite the 60° angle.



$$\frac{x}{\sin 60} = \frac{4}{\sin 30}$$

$$x = \frac{2}{4} \times \frac{\sqrt{3}}{x} \Big/ \frac{1}{2}$$

$$= \boxed{4\sqrt{3}}$$

Problem 6: In triangle ABC, angle C is a right angle, and sin(A) = 3/5. If the hypotenuse is 20, find the length of the side opposite angle A.

$\angle A$	0°	30°	45°	60°	90°
$\sin A$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos A$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan A$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

Problem 7: A ladder 10 feet long leans against a wall and makes a 60° angle with the ground. How high up the wall does the ladder reach?

Problem 8: In triangle ABC, angle C is a right angle, and BC = 12, AC = 5. Find the tangent of angle B.

Problem 9: In triangle ABC, angle C is a right angle, and angle A is 45°. If AC = 7, find the length of AB.

Problem 10: In a triangle, one side measures 6, the adjacent angle is 30°, and the hypotenuse is 12. Find the sine of the angle opposite the side of length 6.

sine of the angle opposite the side of length c.

Maths Circle Senior - Algebra

Saturday, July 5, 2025 11:46 PM

Problem 1:
Solve for all real x:
$(x^2 - 5x + 6) / (x^2 - 4) = 1$

$x^2 - 5x + 6 = x^2 - 4$

$-5x = -10$

$x = 2$

$\frac{x^2-5x+6}{x^2-4} \times (x^2-4) = 1 \times (x^2-4)$

$x \neq \pm 2$

$\therefore$ there are no real solutions.

Problem 2:
Let $f(x) = x^3 - 3x + 1$.
Find the sum of all real solutions to the equation
$f(f(x)) = 1$.

Problem 3:
If $x + 1/x = 4$, find the value of $x^3 + 1/x^3$.

$$x^3 + \frac{1}{x^3}$$

$$= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$$

$$= 4^3 - 3(4)$$

$$= 64 - 12$$

$$= 52$$

Problem 4:

Let a and b be real numbers such that $a + b = 7$ and $ab = 10$. Find the value of $a^2 + b^2$.

a^2+b^2

$=(a+b)^2 - 2ab$

$= 7^2 - 2(10)$

$= 49 - 20$

$= 29$

Problem 5:
Solve for real x:
$\log_2(x^2 - 3x) = 3$

Problem 6:
Let $f(x) = 2x + 3$ and $g(x) = x^2$. Find all x such that $f(g(x)) = g(f(x))$.

Problem 7:
Suppose x and y are real numbers such that $x + y = 5$ and $x^2 + y^2 = 13$. Find $x^4 + y^4$.

Problem 8:
Find the number of real solutions to the equation:
$(x^2 - 2x + 2)(x^2 + 2x + 2) = 1$

Problem 9:

Problem 9:
Let r and s be the roots of the equation $x^2 - 5x + 6 = 0$.
Find the value of $r^3 + s^3$.

Problem 10:
Solve for all real x:
2^(x + 1) + 2^(x - 1) = 20

Maths Circle Senior - Combinatorics

Saturday, March 29, 2025 11:45 PM

1. Grid Paths with Restrictions

A robot starts at (0,0) and moves to (6,4), only moving right or up. However, it cannot pass through the point (3,2). How many paths can the robot take?

$(0,0) \rightarrow (6,4)$ $\quad 210 - 100 = \boxed{110}$

$10C_4 = 210$

$(0,0) \rightarrow (3,2)$

$5C_2 = 10$

$(3,2) \rightarrow (6,4)$ $\rightarrow x = 100$

$5C_2 = 10$

2. Counting Arrangements with Restrictions

Seven students, including Alice and Bob, are to be seated in a row. If Alice must be seated to the left of Bob, how many different seating arrangements are possible?

exactly half of the arrangements

$\frac{7!}{2} = \boxed{2520}$

3. Counting Subsets with a Sum Condition

How many subsets of {1,2,3,4,5,6,7,8,9} have a sum that is divisible by 3?

4. Arranging People in a Circle

Eight people are seated in a circle. In how many ways can they be arranged if two specific people must sit opposite each other?

$6! = 720$

$(n-1)!$

5. Dividing Students into Groups

Ten students are to be split into three groups, such that each group contains at least one student. How many ways can this be done if the groups are unlabeled?

.

6. Digit Sum Problem

How many four-digit numbers have digits that sum to 10?

7. String Coloring Problem

A binary string of length 10 consists of only 0s and 1s. How many such strings have no two consecutive 1s?

8. Balls and Bins with Restrictions

Ten identical balls are placed into four distinct bins. How many ways can this be done if no bin can be empty?

9. Counting Paths with a Forbidden Move

A robot moves from (0,0) to (5,5), taking steps of (1,0), (0,1), or (1,1) (a diagonal step). However, the robot can never take two consecutive diagonal steps. How many valid paths exist?

10. Grid Coloring

A 3 × 3 grid is to be colored using 3 colors such that no two adjacent squares have the same color. How many valid colorings exist?

Maths Circle Senior - Combinatorics

Saturday, January 4, 2025 7:41 PM

Problem 1: There are five different teacups and three different tea saucers in the "Tea Party" store. How many ways are there to buy a cup and a saucer?

15

Problem 2: There are five different teacups, three saucers, and four teaspoons in the "Tea Party" store. How many ways are there to buy two items with

different names?

Case 1: Cup and Saucer

$5 \times 3 = 15$

$15 + 20 + 12 = 47$

Case 2: Cup and Spoon

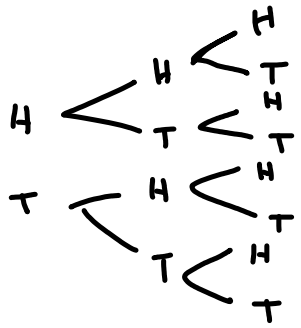
$5 \times 4 = 20$

Case 3: Saucer and spoon

$4 \times 3 = 12$

Problem 3: We toss a coin three times. How many different sequences of heads and tails can we obtain?

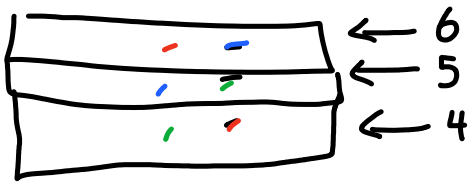
coin → heads or tails



$2 \times 2 \times 2 = 8$

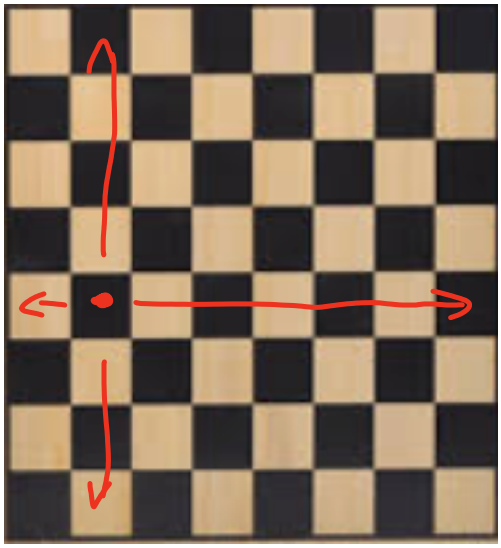
Problem 4: How many ways are there to sew one three-colored flag with three horizontal strips of equal

height if we have pieces of fabric of six colors? We can distinguish the top of the flag from the bottom



$6 \times 5 \times 4 = 120$

Problem 5: How many ways are there to put one white and one black rook on a chessboard so that they do not attack each other?

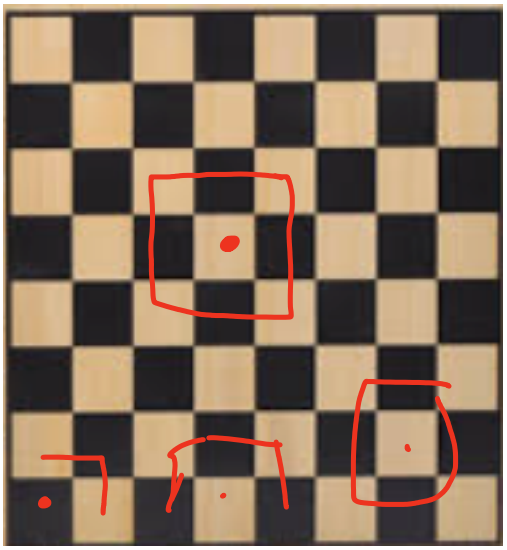


First rook: 64 options

Second rook: $64 - 15 = 49$ options

64×49

Problem 6: How many ways are there to put one white and one black king on a chessboard so that they do not attack each other?



Case 1: Corner

64 − 4 = 60

60 × 4

Case 2: Side

24 × 58

Case 3: Central

36 × 55

60×4 + 24×58 + 36×55

Problem 7: How many three-digit numbers can be written using the digits 1, 2, and 3 (without repetitions) in some order?

_ _ _

3 × 2 × 1 = 6

Problem 8: How many ways can I rearrange the letters in the word "Vector"?

Problem 9: How many ways can I rearrange the letters in the word "Trust"?

Problem 10: How many ways can I rearrange the letters in the word "Caravan"?

11) MATHEMATICAL

$$\frac{12!}{3! \times 2! \times 2!}$$

word → n letters, m times

$$\frac{n!}{\_}$$

ans: 19958400

$\frac{}{m!}$

Problem 12: There are 20 towns in a certain country, and every pair of them is connected by an air route. How many air routes are there?

190

Problem 13: How many diagonals are there in a convex n-gon, where n is the number of sides/vertices?

$$\frac{(n)(n-3)}{2}$$

Homework:

Problem 1: Jiwei and Hari entered a race. Hari finished the race in 4/5 of the time it took Jiwei to finish. The next time that they raced the same distance, Jiwei increased his average speed from the first race by x%, while Hari maintained the same average speed as in the first race. In this second race, Hari finished the race in the same amount of time that it took Jiwei to finish. Find x.

Problem 2: Each of the four digits of the integer 2024 is even. How many integers between 1000 and 9999, inclusive, have the property that all four of their digits are even?

Problem 3: In a survey, 100 students were asked if they like lentils and were also asked if they like chickpeas. A total of 68 students like lentils. A total of 53 like chickpeas. A total of 6 like neither lentils nor chickpeas. How many of the 100 students like both lentils and chickpeas?

Problem 4: Ellie's drawer of hair clips contains 4 red clips, 5 blue clips, and 7 green clips. Each morning, she randomly chooses one hair clip to wear for the day. She returns this clip to the drawer each evening. One morning, Kyne removes k hairclips before Ellie can make her daily selection. As a result, the probability that Ellie chooses a red clip is doubled. Which of the following is a possible value of k?

Maths Circle Senior - Divisibility

Saturday, April 26, 2025 11:04 PM

Problem 1: Find the smallest positive integer that leaves a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, and a remainder of 3 when divided by 5.

$n > 5$

$n \equiv 1 \bmod 2$

$n \equiv 2 \bmod 3$ ⟷ $5k+3 \equiv 2 \bmod 3$

$n \equiv 3 \bmod 5$

$5k \equiv -1 \bmod 3$

$n = 5k + 3$

$2k \equiv -1 \bmod 3$

$n = 5(3m+1)+3$

$2k \equiv 2 \bmod 3$

$= 15m + 8$

$k \equiv 1 \bmod 3$

$k = 3m + 1$

for $m = 1$:

$15 + 8 = 23$

$23 \equiv 1 \bmod 2$

$23 \equiv 2 \bmod 3$

$23 \equiv 3 \bmod 5$

Problem 2: Determine how many positive integers less than 2000 are divisible by neither 5 nor 7.

divisible by 5 → 399

divisible by 7 → 285

divisible by 5 & 7 (35) → 57

$1999 - (399 + 285 - 57) =$ 1372 integers

*

"principle of inclusion-exclusion"

Problem 3: A five-digit number is divisible by 9 and ends with two 7's. What is the smallest such number?

1 0 3 7 7

Problem 4: Find the number of positive integers less than 10,000 that are divisible by 3 or 7 but not divisible by 5.

divisible by 3 : 3333

divisible by 7 : 1428

divisible by 3 and 7 (21) : 476

divisible by 7 or 3 and 5 :

↳ 15 : 666
35 : 285

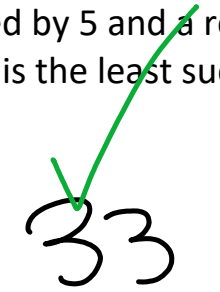
} 666+285−95

105 : 95 = 856

4285 − 856
= 3429

3333 + 1428 − 476
= 4285

Problem 5: A number leaves a remainder of 3 when divided by 5 and a remainder of 5 when divided by 7. What is the least such positive integer?



Problem 6: What is the sum of all positive integers less than 1000 that are divisible by 8 and leave a remainder of 1 when divided by 3?

1st step: find a pattern

2nd step: find a sequence / series

arithmetic progression:

$a, a+d, a+2d, \ldots\ldots a+(n-1)d$

arithmetic series (sum):

$\frac{n}{2}(2a+(n-1)d)$

↳ $\frac{n}{2}(a + a+(n-1)d)$

↳ $\frac{n}{2}(a + a_n)$

Problem 7: Find the largest three-digit integer that is divisible by 17 and whose digits are all distinct.

Problem 8: How many integers between 1000 and 9999 are divisible by 11 and have all digits distinct?

Problem 9: Determine the smallest positive integer n such that both n and n+1 are products of exactly two distinct prime numbers.

Problem 10: Let N be the smallest positive integer such that 2N ends with the digits 444. Find N.

Maths Circle Senior - Geometry

Sunday, August 10, 2025 9:30 AM

Problem 1:
A triangle has sides 13, 14, and 15. Find its area.

$s = \frac{(13+14+15)}{2} = 21$

$A = \sqrt{(21)(8)(7)(6)}$

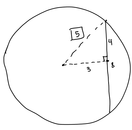
$= 84$ ✓

Problem 2:
In a regular hexagon with side length 6, find the distance between two opposite vertices.

$6+6\sqrt{3}$

✓

Problem 3:
A circle has a chord of length 8 that is 3 units away from the center. Find the radius of the circle.



✓

Problem 4:
In a right triangle, the altitude to the hypotenuse is 6, and one leg is 8. Find the other leg.

$\frac{8\cdot 6}{2} = \frac{b\cdot c}{2}$

$\frac{(8)(6)}{c} = 6 \rightarrow c = \frac{48}{6}$

$8^2+b^2=c^2$

$64+b^2=\frac{144}{9}$ ✓

$\frac{72^2}{9} = 64$

$b = \sqrt{\frac{144}{7}}$

$b = \frac{24}{\sqrt{7}}$

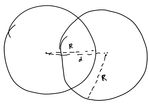
Problem 5:
A square has side length 10. A circle is inscribed in it. Find the area of the region inside the square but outside the circle.

$100 - 25\pi$

or

21.5

Problem 6:
Two circles of radius 5 overlap so that the distance between their centers is 6. Find the area of their intersection.



$A_{wedge} = 2R^2\cos^{-1}\left(\frac{3}{5}\right) - \frac{1}{2}\left(10\sqrt{4^2}\right)$

$= 22.57$

Problem 7:
A triangle has vertices A(0, 0), B(6, 0), and C(2, 5). Find the length of the median from A.

Problem 8:
In an acute triangle ABC, AB = 13, BC = 14, and AC = 15. Find the length of the altitude from B.

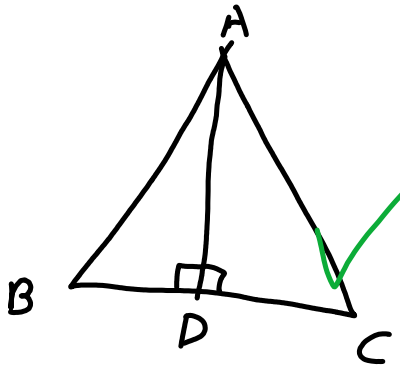
Problem 9:
A regular tetrahedron has edges of length 1. Find its volume.

Problem 10:
A cylinder has radius r and height h such that its total surface area equals its volume. Find h in terms of r.

Maths Circle Senior - Geometry

Saturday, April 5, 2025 9:45 PM

Problem 1: In triangle ABC, AB = 13, BC = 14, and AC = 15. Let D be the foot of the altitude from A to BC. What is the length of AD?



Heron's formula

s = semi-perimeter : 21

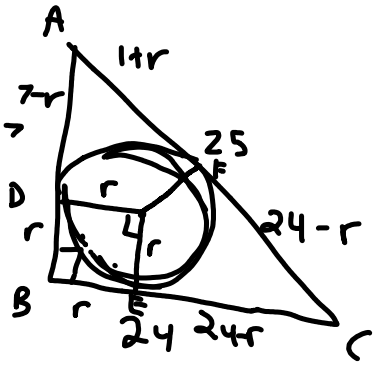
$A = \sqrt{s(s-AB)(s-BC)(s-AC)}$

$= 84$

$A = \frac{1}{2}bh$

$h = 12$

Problem 2: A circle is inscribed in a right triangle with legs 7 and 24. What is the radius of the circle?

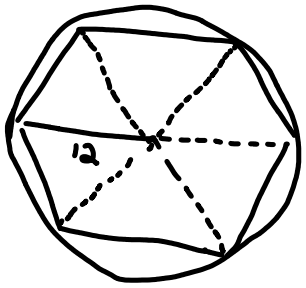


$7 - r = 1 + r$

$2r = 6$

$r = 3$

Problem 3: A regular hexagon is inscribed in a circle of radius 12. What is the area of the hexagon?

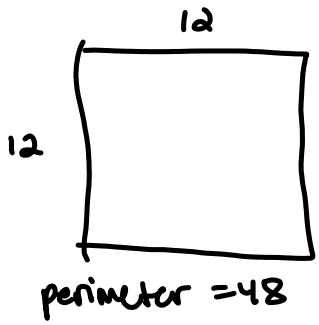


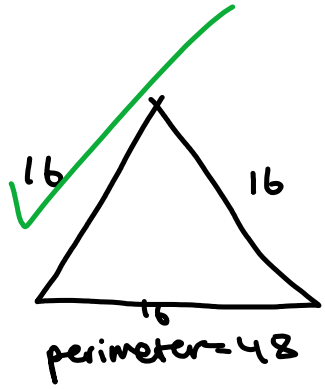
$\frac{\sqrt{3}}{4} s^2 \times 6$

$= \frac{\sqrt{3}}{4} (12)^2 \times 6$

$= 216\sqrt{3}$

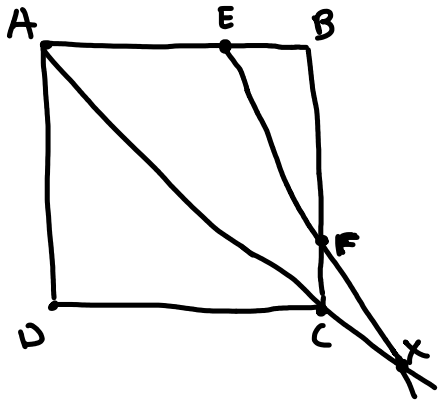
Problem 4: A square and an equilateral triangle have equal perimeters. If the square has side length 12, what is the area of the triangle?





$\frac{\sqrt{3}}{4} s^2 = \frac{\sqrt{3}}{4} (16)^2 = 64\sqrt{3}$

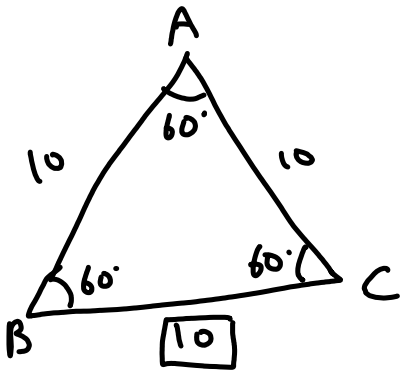
Problem 5: Let ABCD be a square. Points E and F lie on sides AB and BC respectively such that AE = BF = 1. Segment EF intersects diagonal AC at point X. What is the ratio of the area of triangle AEX to the area of triangle CFX?



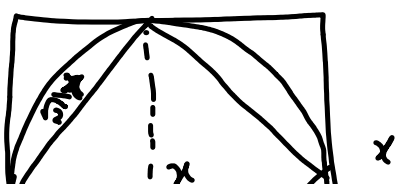
Maths Circle Senior - Geometry

Sunday, June 29, 2025 12:00 AM

Problem 1:
In triangle ABC, angle A is 60 degrees and AB = AC = 10. Find the length of BC.



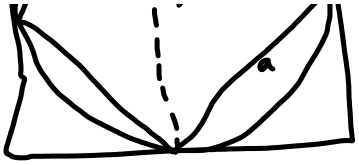
Problem 2:
A circle is inscribed in a square. A second square is inscribed in that circle. What is the ratio of the area of the smaller square to the larger square?



area of large square $= x^2$

diameter of circle $= x$

$a^2 + a^2 = x^2$



$2a^2 = x^2$

$a = \sqrt{\frac{x^2}{2}} = \frac{x}{\sqrt{2}}$

side length of smaller square = $\frac{x}{\sqrt{2}}$

area " " " $\frac{x^2}{2}$

$\frac{x^2}{2} : x^2 = \frac{1}{2} : 1$

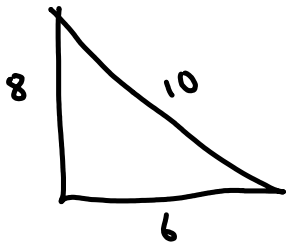
$\boxed{\frac{1}{2}}$

Problem 3:
A triangle has side lengths 13, 14, and 15. Find the area of the triangle.

$a = 13, b = 14, c = 15, s = 21$

$A = \sqrt{s(s-a)(s-b)(s-c)} = \boxed{84}$ ✓

Problem 4:
Let A(0, 0), B(6, 0), and C(0, 8) be the vertices of triangle ABC. Find the radius of the circle inscribed in triangle ABC.



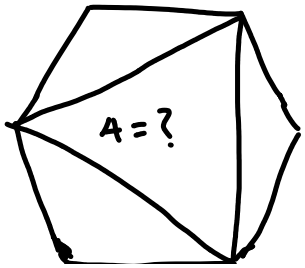
radius = 2

$A = \frac{1}{2} \times 8 \times 6 = 24$ ✓

$s = \frac{6+8+10}{2} = 12$

$\text{inradius} = \frac{A}{s} = \boxed{2}$

Problem 5:
In a regular hexagon, what is the ratio of the area of an equilateral triangle formed by connecting every other vertex to the area of the hexagon?



1 : 2

Problem 6:
A right circular cone has height 12 and radius 5. A smaller cone is sliced from the top, similar to the original, such that its height is 6. What is the volume of the smaller cone?

Problem 7:
A trapezoid has bases of lengths 10 and 20, and height 6. A line parallel to the bases divides the trapezoid into two regions of equal area. How far from the longer base is the line?

Problem 8:
A circle has center O and radius r. A point A lies outside the circle such that the length of the tangent from A to the circle is 12 and the distance from A to O is 13. Find r.

Problem 9:
A square has side length 10. A circle is inscribed in the square, and a smaller square is inscribed in the circle. What is the area of the smaller square?

Problem 10:
In a triangle, the medians have lengths 5, 6, and 7. Find the area of the triangle.

Maths Circle Senior - Geometry

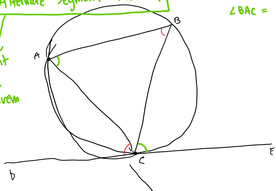
Sunday, August 17, 2025 9:33 AM

Alternate Segment Theorem

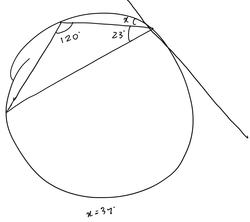
Tangent Chord Theorem

$\angle ADC = \angle ACD$

$\angle BAC = \angle BCE$

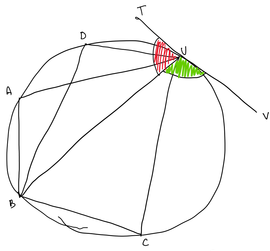


1)



$x = 37^\circ$

2)



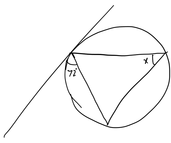
Which of these is equal to $\angle VUB$?

a) $\angle VBC$

b) $\angle VCB$

c) $\angle VAB$

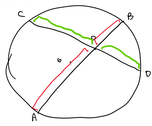
3)



Find x $\quad x = 72^\circ$

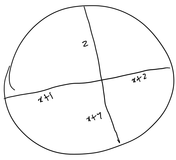
Intersecting Chord Theorem

Power of a Point Theorem



$PA \times PB = PC \times PD$

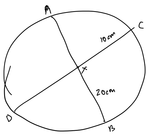
4)



Find x.

$x = 3$

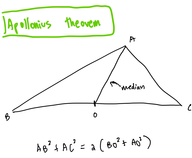
5)



$AX : XB = 2 : 5$

$DX = ? = 16$

Apollonius Theorem



$AB^2 + AC^2 = 2(BO^2 + AO^2)$

6) Find the length of the median for a triangle with sides 5cm, 10cm, and 12cm, the median meeting the 12cm side.

$\sqrt{\frac{93}{2}}$

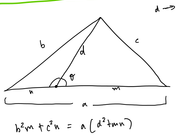
$\therefore 6.84$ cm

7) Find the value of the 3 sides if they are $4x$, $9x$, and $7x$ and the median meeting the $7x$ side is 5 cm.

$x = \sqrt{7.8}$

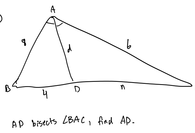
$\therefore 9.48,\ 11.92,\ 12.64$

Stewart's Theorem



$b^2 m + c^2 n = a(d^2 + mn)$

8)



Area of a triangle $= \frac{1}{2} ab \sin\theta$

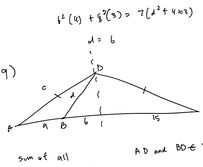
AD bisects $\angle BAC$; Find AD.

$\frac{n}{4} = \frac{\frac{1}{2} \cdot 6 \cdot x \sin\theta}{\frac{1}{2} \cdot 9 \cdot x \sin\theta}$

$n = 3$

$x^2(9) + 6^2(3) = 7(x^2 + 4 \cdot 3)$

9)



Sum of all possible perimeters is 5, find 5

AD and $BD \in \mathbb{Z}^+$

$30x^2 + 2(x+x)x + 30 = 9c^2 + 21c^2$

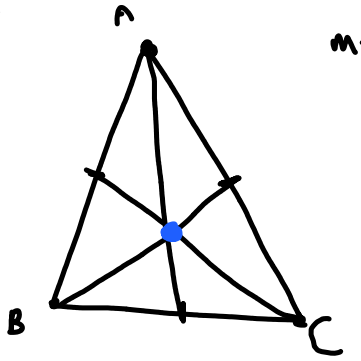
$d^2 - c^2 = 189$

$(d+c)(d-c) = 189$

Maths Circle Senior - Geometry

Friday, December 13, 2024 5:24 PM

Median and centroid of a triangle:

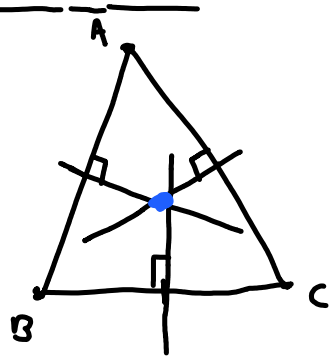


median: vertex to midpoint of the opposite side

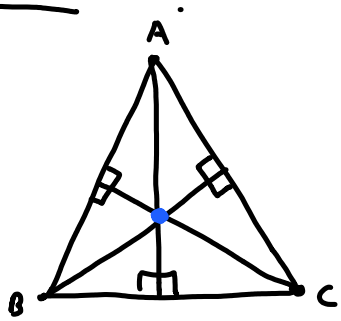
centroid: intersection of all medians

Perpendicular bisector and circumcenter:

bi = 2

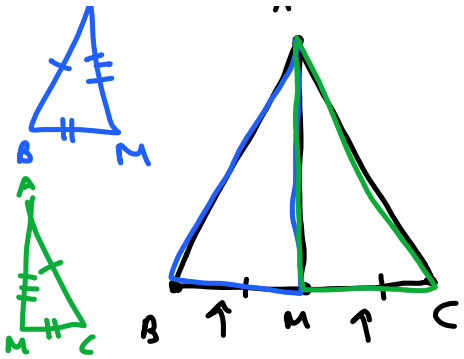


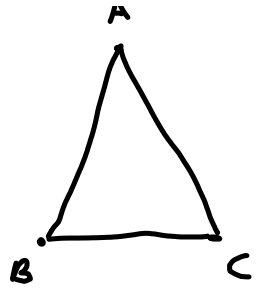
Altitude and orthocenter:



Problem 1: Prove that the length of median AM in triangle ABC is greater than (AB+ AC - BC)/2.

TRIANGLE INEQUALITY





$AB+AC > BC \quad |AB-AC| < B$

$AC+BC > AB \quad |AC-BC| < A$

$AB+BC > AC \quad |AB-BC| <$

$AM > AB-BM$

$+ \quad AM > AC-CM$

$2AM > AB+AC-BC$

$BM = CM$

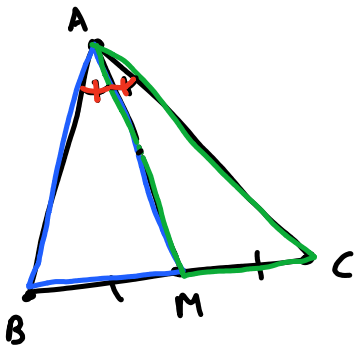
$-BM-CM$

$-(BM+CM)$

$$AM > \frac{AB+AC-BC}{2}$$

Problem 2: In triangle ABC the median AM is longer than half of BC. Prove that angle BAC is acute.

$\angle BAC < 90^\circ$



$$AM > \frac{BC}{2}$$

$AM > BM$

$AM > CM$

In $\triangle ABM$:

$\angle BAM + \angle AMB + \angle MBA = 180^\circ$

In $\triangle ACM$:

$\angle AMC + \angle MCA + \angle CAM = 180^\circ$

In $\triangle ABC$:

$\angle ABC + \angle BCA + \angle CAB = 180^\circ$

$\angle BAM + \angle CAM = \angle CAB$

$$AM = \sqrt{\frac{2AB^2+2AC^2-BC^2}{4}}$$

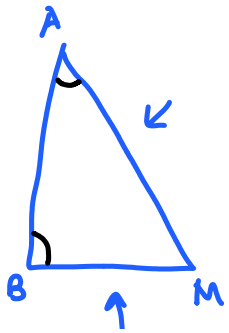
Sukdeb's solution

$BAM < ABC$

$CAM < ACB$

$BAC < ABC + ACB$

If $BAC > 90^\circ$:




C
AB
∠AC

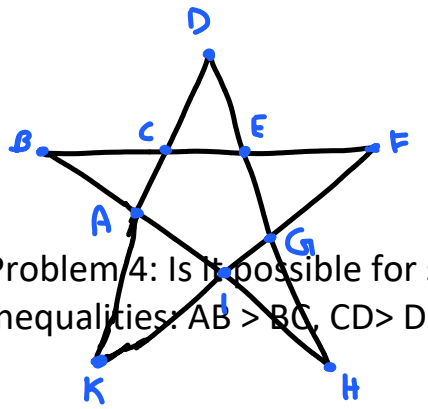
Assume [illegible]

$\angle ABC + \angle ACB > 90^\circ$

$\angle ABC + \angle ACB + \angle BAC > 180^\circ$ ✗ not true

✗ contradiction

Problem 3: ABCD is a convex quadrilateral and AB+ BD < AC



Problem 4: Is it possible for some five-pointed star ABCDEFGHIK to satisfy the inequalities: AB > BC, CD> DE, EF > FG, GH > HI, IK >KA?

Problem 5: A point is selected on each side of a square. Prove that the perimeter of the quadrilateral formed by these points is no less than twice the length of the square's diagonal.

Maths Circle Senior - Graphs (Nov 23)

Saturday, November 23, 2024 8:47 PM

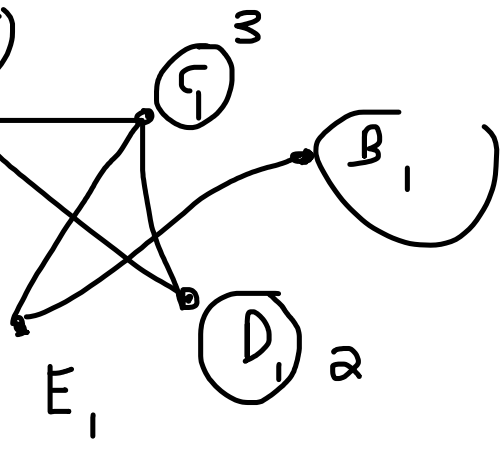
What is a graph?
Collection of vertices joined by edges

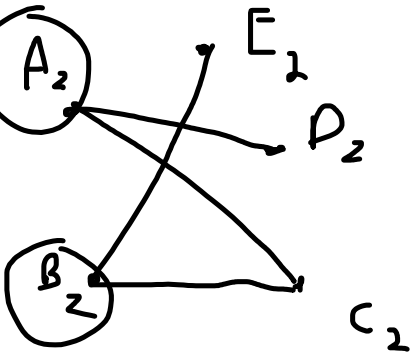
What do we mean by the degree of a vertex?
Number of edges that are connected to it

What is isomorphism?

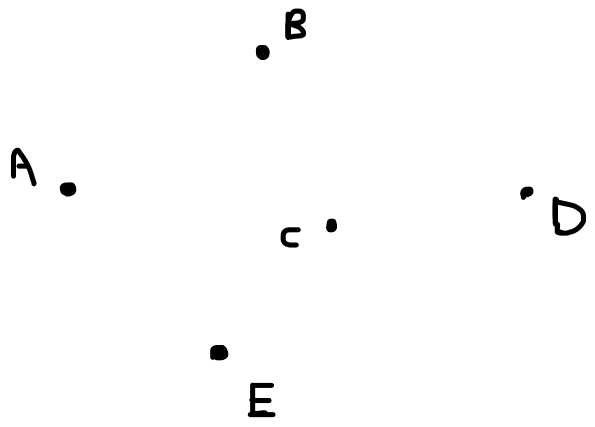
- Same number of vertices
- Each vertex is connected to the same number of vertices
- Connected to the exact same vertices

Problem 1: Prove that there does not exist a graph with 5 vertices with


3
C_1
B_1
D_1
2
E_1

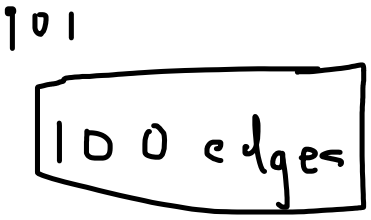

A_2
E_2
D_2
B_2
C_2

degrees equal to 4, 4, 4, 4, and 2.

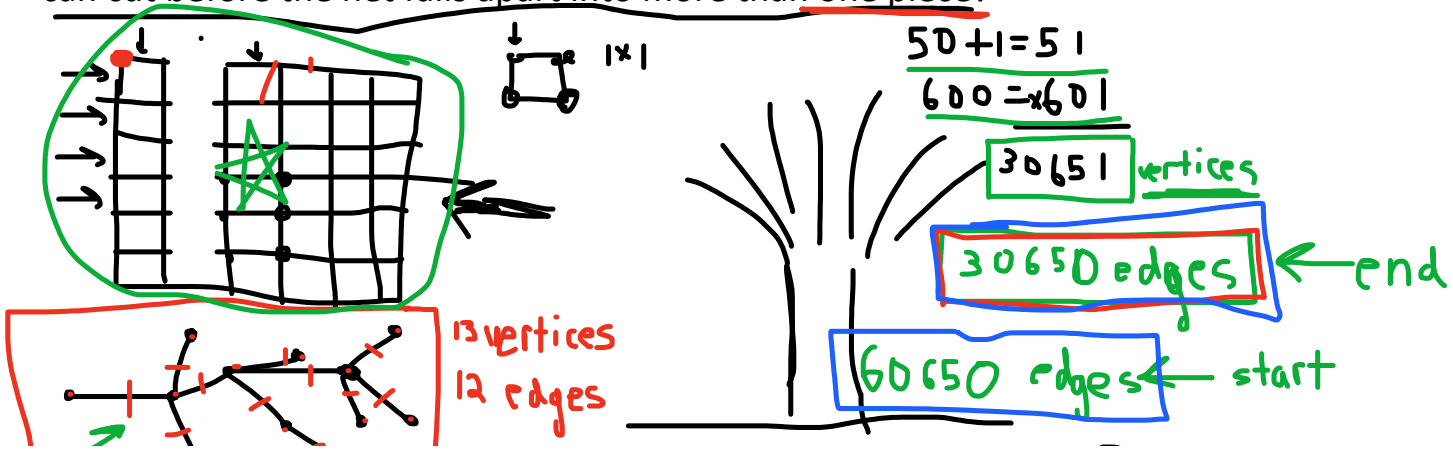


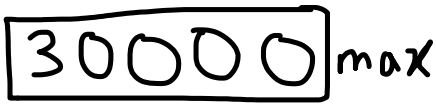
Problem 2: There are 101 towns in Forestland. Some of them are connected by roads, and each pair of towns is connected by one and only one simple path. How many roads are there?

Pendant vertex: vertex with degree of 1



Problem 3: A volleyball net has the form of a rectangular lattice with dimensions 50 x 600. What is the maximum number of unit strings you can cut before the net falls apart into more than one piece?

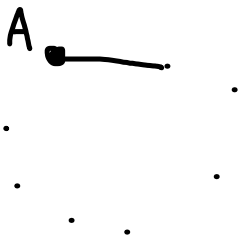




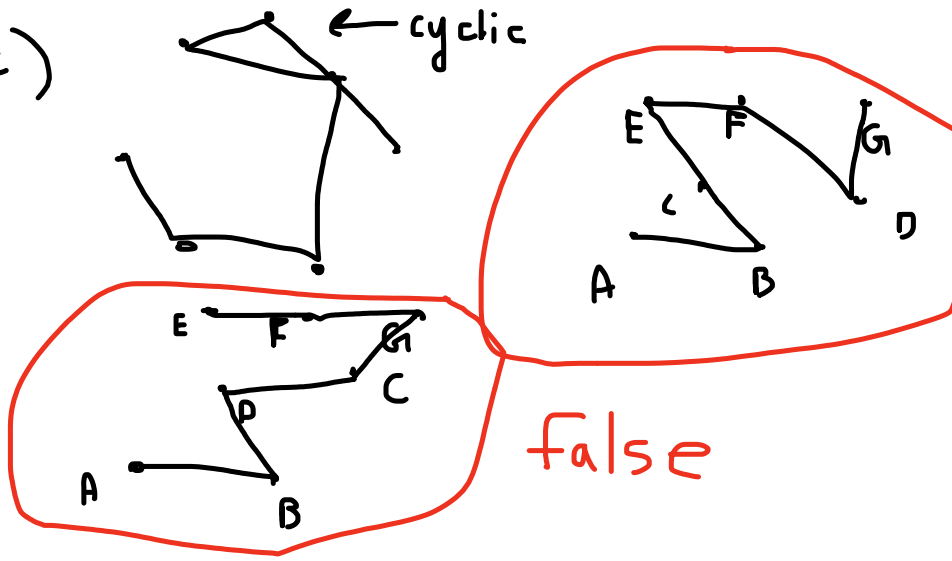
Problem 4: Is it true that two graphs must be isomorphic, if
a) they both have 10 vertices and the degree of each equals 9?
b) they both have 8 vertices and the degree of each equals 3?
c) they are both connected, without cycles, and have 6 edges?

a) True

b) False



c)



Problem 5: There are 100 towns in a country and some of them are connected by airlines. It is known that one can reach every town from any other (perhaps with several intermediate stops). Prove that you can fly around the country and visit all the towns making no more than a) 198 flights; b) 196 flights.

Problem 6: There are 7 lakes in Lakeland. They are connected by 10 canals so that one can swim through the

canals from any lake to any other. How many islands are there in Lakeland?

Problem 7: There are 20 points inside a square. They are connected by nonintersecting segments with each other and with the vertices of the square, in such a way that the square is dissected into triangles. How many triangles do we have?

Homework:

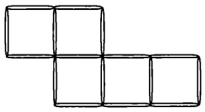
Graphs Problem 1: There are 100 circles forming a connected figure on the plane. Prove that this figure can be drawn without lifting the pencil

off the paper or drawing any part of any circle twice.

Graphs Problem 2: There are 50 scientists at a conference and each of them is acquainted with at least 25 of the others. Prove that there are four of them who can be seated at a round table so that each of them has two acquaintances for neighbors.

Miscellaneous Problem 1: Is it possible to arrange six long round pencils so that each of them touches all the others?

Miscellaneous Problem 2: Matches are arranged to form the figure shown on the right. Move two matches to change this figure into four squares with sides equal in length to one match.



Miscellaneous Problem 3: Three tablespoons of milk from a glass of milk are poured into a glass of tea, and the liquid is thoroughly mixed. Then three tablespoons of this mixture are poured back into the glass of milk. Which is greater now: the percentage of milk in the tea or the percentage of tea in the milk?

Maths Circle Senior - Integers

Saturday, January 18, 2025 11:41 PM

Problem 1: Find all natural numbers equal to the sum of the factorials of their digits.

1, 2, 145, 40585

max possible digits = 7

$0! = 1$ $n! = n \times n-1 \times n-2 \times \dots \times 1$

n-digit number : $9! \times n$

n-digit number : 10^{n-1}

$n \cdot 9! \geq 10^{n-1} \rightarrow n = 7$ $2540040 > 1000000$

$n = 8$ $2903040 < 10000000$

$0! = 1$
$1! = 1$
$2! = 2$
$3! = 6$
$4! = 24$
$5! = 120$
$6! = 720$
$7! = 5040$
$8! = 40320$
$9! = 362880$

single digit numbers : 2

2-digit numbers : 0

3-digit numbers : 1

4-digit numbers : 0

5-digit numbers : 1

Problem 2: Prove that among any 39 consecutive natural numbers one can find a number, the sum of whose digits is divisible by 11.

Let sum of digits $\rightarrow S(n)$

$S(n) \equiv 0 \mod 11$

Problem 3: Does there exist a two-digit number whose square ends in the same two digits, but in the reverse order?

Problem 4: Find all natural numbers n such that the number $n^n + 1$ is prime and has no more than 19 digits.

Problem 5: The fourth power of a natural number

is written with the digits 0, 1, 4, 6, 7, 9 in some order. Find this number.

Problem 6: Integers a, b, and c are given, and it is known that ax2 + bx+ c is divisible by 5 for any integer x. Prove that a, b, and c are themselves divisible by 5.

Maths Circle Senior - Invariants

Saturday, February 8, 2025 10:20 PM

What is an invariant?

Quantity that doesn't change

even when other values change

↳ can help us solve very hard

looking problems in simple steps

SPOT THE INVARIANT

Problem 1: The numbers 1, 2, 3, ... , 19, 20 are written on a blackboard. It is allowed to erase any two numbers a and b and write the new number a + b - 1. What number will be on the blackboard after 19 such operations?

$S + a + b \longrightarrow S + a + b - 1$

sum ↓ 1 each time

−19

$1+2+3+\dots+20 = \frac{20 \times 21}{2} = 210$

210

↓

−19

↓

191

Problem 2: The numbers 1, 2, ... , 20 are written on a blackboard. It is permitted to erase any two numbers a and b and write the new number ab + a + b. Which number can be on the blackboard after 19 such operations?

$ab+a+b = (a+1)(b+1)-1$

$P_{old} = (1+1)(2+1)(3+1)\ldots\ldots(20+1) = 21!$

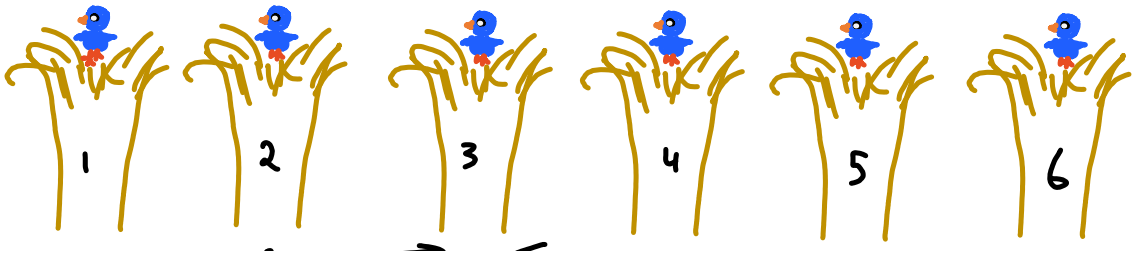
$(ab+a+b)+1 = (a+1)(b+1)$

$P_{new} = \dfrac{P_{old}}{(a+1)(b+1)} \cdot (a+1)(b+1) = P_{old}$

$(x+1) = P = 21!$

$x = 21! - 1$

Problem 3: There are six sparrows sitting on six trees, one sparrow on each tree. The trees stand in a row, with 10 meters between any two neighboring trees. If a sparrow flies from one tree to another, then at the same time some other sparrow flies from some tree to another the same distance away, but in the opposite direction. Is it possible for all the sparrows to gather on one tree? What if there are seven trees· and seven sparrows?



$S = 21$ 

↳ invariant

$6k \neq 21$

Problem 4: There are 13 gray, 15 brown, and 17 red chameleons on Chromatic Island. When two chameleons of different colors meet they both change their color to the third one (for instance, gray and brown both become red). Is it possible that after some time all the chameleons on the island are the same color?

gray → a → number of chameleons

brown → b → " " " "

red → c → " " " "

$(a,b,c) \longrightarrow (a-1, b-1, c+2)$ OR $(a-1, b+2, c-1)$ (0) (3)

OR $(a+2, b-1, c-1)$ (3)

$a-b = -2$

$a-b = 0$ $\quad -2 \to 0$ by subtracting 3?

∴ it is not possible.

Problem 5: In an 8 x 8 table one of the boxes is colored black and all the others are white. Prove that one cannot make all the boxes white by recoloring the rows and columns. "Recoloring" is the operation of changing the color of all the boxes in a row or in a column.

Problem 6: Solve the same problem for an 8 x 8 table if initially all four corner boxes are black and all the others are white.

Maths Circle Senior - Logarithms

Saturday, July 12, 2025 11:42 PM

Problem 1:
If $\log_2(x) = 5$, what is the value of x?

$x = 2^5$

$= 32$

$\log_a a = c \longrightarrow a = b^c$

base

$\log_a b \longrightarrow \log_{10} b$

$\ln b \longrightarrow \log_e b$

↳ natural logarithm

$e \rightarrow$ euler's number

$\approx 2.71828\ldots\ldots$

$\log_b(M \times N) = \log_b M + \log_b N$

$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

$\log_b(M^k) = k \log_b M$

$\log_b 1 = 0 \rightarrow b^0 = 1$

$\log_b(b^k) = k \rightarrow b^k = b^k$

$b^{\log_b k} = k$

$\rightarrow k \log_b b = k \times 1 = k$

$M, N, b > 0;\ M, n, k \in \mathbb{R},\ b \neq 1$

Problem 2:
Simplify: $\log_3(81) - \log_3(3)$

$= \log_3\left(\frac{81}{3}\right)$

$= \log_3(27) = x$

$3^x = 27$

$3^x = 27$

$x = 3$

$3^a = 81$ $\quad 3^b = 3$

$a = 4$ $\quad b = 1$

$= 4 - 1$

$= 3$

Problem 3:
Evaluate: $\log_5(25) + \log_5(1/5)$

$= \log_5\left(25 \times \frac{1}{5}\right)$

$= \log_5 5$

$= 1$

Problem 4:
If $\log_4(x) = 3$, express $\log_2(x)$ in terms of integers.

↳ $x = 4^3$ ↳ $x = 2^t$

$= 2^1 \times 2^3$

$= 2^4$

$\log_2 x = 4$

Problem 5:
Solve for x: $\log_2(x^2 - 1) = 3$

$x^2 - 1 = 2^3 = 8$

$x^2 = 9$

$x = \pm 3$

Problem 6:
If $\log_{10}(x) + \log_{10}(x - 9) = 2$, find x.

$\log_{10}(x(x-9)) = 2$

$x^2 - 9x = 100$

$x^2 - 9x - 100 = 0$

$x = \frac{9 \pm \sqrt{481}}{2}$

$\frac{9 - \sqrt{481}}{2}$ is indeterminable

$x = \frac{9 + \sqrt{481}}{2}$

Problem 7:
If $\log_a(b) = 2$ and $\log_b(c) = 3$, find $\log_a(c)$

$b = a^2$ $\quad c = b^3$

$= (a^2)^3$

$= a^6$

$c = a^6$

$\log_a c = 6$

Problem 8:
Given that $\log_7(x) = a$ and $\log_7(y) = b$, express $\log_7(x^2 / \sqrt{y})$ in terms of a and b.

$\sqrt{y} = y^{\frac{1}{2}}$

$\log_7\left(\frac{x^2}{\sqrt{y}}\right)$

$= \log_7(x^2) - \log_7(y^{\frac{1}{2}})$

$= 2\log_7(x) - \frac{1}{2}\log_7(y)$

$= 2a - \frac{1}{2}b$

Problem 9:
Solve for x: $\log_3(x) + \log_3(x - 2) = 1$

$\log_3(x(x-2)) = 1$

$x^2 - 2x = 3$

$x^2 - 2x - 3 = 0$

$x = 3,\ -1$ inadmissible

$x = 3$

Problem 10:
If $\log_2(x) = 3$ and $\log_4(y) = 2$, what is the value of $\log_2(xy)$?

Maths Circle Senior - Miscellaneous

Sunday, July 27, 2025 9:32 AM

- any new principle/theorems
- number theory: quadratic reciprocity

Problem 1:
What is the remainder when 7^2025 is divided by 100?

$7^1 = 07$

$7^2 = 49$

$7^3 = 43$

$7^4 = 01$

$7^5 = 07$

$\vdots$

$2025 \equiv 1 \bmod 4$

remainder is $\boxed{7}$

Problem 2:
A square and a circle have the same area. What is the ratio of the side length of the square to the radius of the circle?

square $\rightarrow x^2$

circle $\rightarrow \pi y^2$

$\pi y^2 = x^2$

$\frac{x^2}{y^2} = \pi$

$\frac{x}{y} = \boxed{\sqrt{\pi}}$

Problem 3:
Let x be a real number such that x + 1/x = 4. What is the value of x^2 + 1/x^2?

$\left(x + \frac{1}{x}\right)^2 = 16$

$x^2 + \frac{1}{x^2} + 2 = 16$

$x^2 + \frac{1}{x^2} = \boxed{14}$

Problem 4:
How many 4-digit numbers have digits that are strictly increasing from left to right?

w x y z

choose

$^9C_4 = \boxed{126}$

Problem 5:
A box contains 4 red balls, 3 blue balls, and 2 green balls. Two balls are drawn without replacement. What is the probability both are the same color?

$^9C_2 = 36$

R: $^4C_2 = 6$

B: $^3C_2 = 3$

G: 1

$\frac{6+3+1}{36} = \frac{10^5}{36_{18}} = \boxed{\frac{5}{18}}$

Problem 6:
A function f satisfies f(x + 1) = 2f(x) for all real x, and f(0) = 3. What is f(5)?

6, 12, 24, 48, $\boxed{96}$

Problem 7:
The sum of the interior angles of a regular polygon is 1440 degrees. How many sides does it have?

$180(n-2) = 1440$

Problem 8:
In how many ways can 5 identical balls be distributed into 3 distinct boxes so that no box is empty?

Problem 9:
Let a and b be positive integers such that the least common multiple of a and b is 84 and the greatest common divisor is 6. What is ab?

$LCM \times GCD = 504$

Problem 10:
How many integers between 1 and 1000 inclusive are divisible by neither 2, 3, nor 5?

266

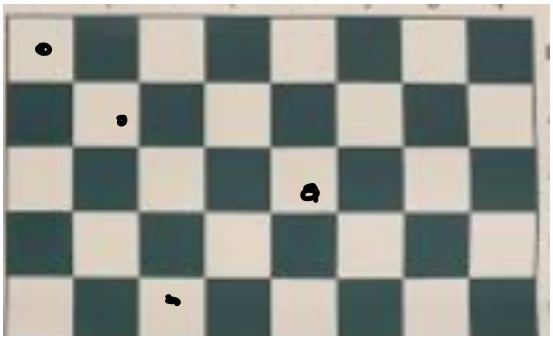
Maths Circle Senior - Miscellaneous Problems

Saturday, March 1, 2025 11:47 PM

Problem 1: Prove that the equality 1/a + 1/b + 1/c + 1/d + 1/e + 1/f = 1 has no solutions in odd natural numbers.

$$\frac{\overbrace{bcdef + acdef + \dots}^{6 \text{ times}}}{abcdef} = \frac{\text{even}}{\text{odd}}$$

★ Problem 2: Eight rooks are placed on a chessboard so that none of them attacks another. Prove that the number of rooks standing on black squares is even.

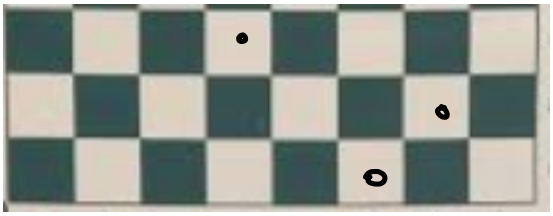


white → w

black → b

w+b = 8

w & b → even



or

w & b → odd

non-attacking squares: $64 - 16w + w^2 - w$

$= 64 - 17w + w^2$

$=$ even (since w is odd)

Problem 3: Is it possible to place 20 red and blue pawns around a circle in such a way that a blue pawn is standing on the point opposite to any red pawn, and no two of the blue pawns are neighbors?

Problem 4: Points A and B are chosen on a straight line. Then 1001 other points are chosen outside segment AB, and these are colored red and blue. Prove that the sum of the distances from A to the red points and from B to the blue points is not equal to the sum of the distances from B to the red points and from A to the blue points.

Problem 5: There are ten pairs of cards with the numbers 0, 0, 1, 1, ... , 8, 8, 9, 9 written on them. Prove that they cannot be laid in a row so that there are exactly n cards between any two cards with equal numbers non them {for all n = 0, 1, ... , 9).

Problem 6: Twenty points, which form a regular 20-gon, are chosen on a circle. Then they are split into ten pairs, and the points in each pair are connected by a chord. Prove that some pair of these chords have the same length.

Maths Circle Senior - Miscellaneous

Saturday, May 24, 2025 11:45 PM

Problem 1:
Find all real solutions to the equation:
$x + \sqrt{(4x + 13)} = 5$

$\sqrt{4x+13} = 5 - x$

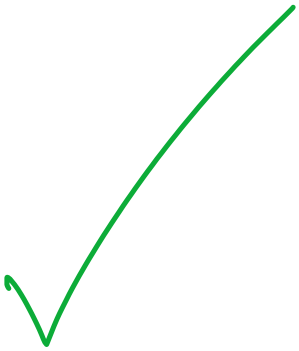
$4x+13 = x^2 - 10x + 25$

$x^2 - 14x + 12 = 0$

$x = 7 \pm \sqrt{37}$

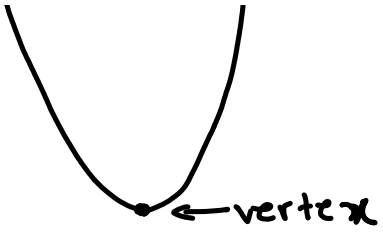
$x \neq 7 + \sqrt{37}$

$\therefore x = 7 - \sqrt{37}$



Problem 2:
Let f(x) = x^2 - 6x + 11. Find the minimum value of f(x).

\ / $a(x-h)^2 + k$



(h, k) ✓

$f(x) = x^2 - 6x + 9 + 2$

$= (x-3)^2 + 2$

(3, 2)

$f'(x) = 2x - 6 = 0$

$x = 3$ ✓

$f(3) = 2$

Problem 3:
How many integers between 1 and 1000 inclusive are divisible by 3 or 5 but not by 15?

divisible by 3 : 333

" " 5 : 200

" " 15 : 66

$333 + 200 - 66 = 467$ ✓

Problem 4:
Let a and b be positive integers such that $a^2 + b^2 = 100$. Find all possible values of a + b.

Problem 5:
Find the number of distinct four-digit numbers that can be

Find the number of distinct four-digit numbers that can be formed using the digits {1, 2, 3, 4, 5} without repetition such that the number is divisible by 5.

Problem 6:
Let $f(x) = x^3 - 3x + 1$. Find all real roots of $f(x) = 0$.

Problem 7:
A 6-sided die is rolled three times. What is the probability that the three rolls are all different?

Problem 8:
If $x + 1/x = 4$, find the value of $x^2 + 1/x^2$.

Problem 9:
In triangle ABC, AB = AC and angle A = 80°. Find the measure of angle B.

Problem 10:
A function f satisfies $f(x + y) = f(x) + f(y)$ and $f(1) = 2$. Find $f(2025)$.

Maths Circle Senior - Modular Arithmetic

Sunday, February 23, 2025 8:21 AM

Problem 1: Find possible values of x if $3^x - x^2$ is divisible by 5 for $\{x \in \mathbb{Z}^+, x \leq 100\}$.

$3^4 \equiv 81 \equiv 1 \bmod 5$

Let x be as $x \equiv s \bmod 20$, $s \leq 20$.

$x \equiv s \bmod 4$, $x \equiv s \bmod 5$

$3^x \equiv 3^s \bmod 20$, $x^2 \equiv s^2 \bmod 20$

$3^x - x^2 \equiv 3^s - s^2 \equiv 0 \bmod 20$

$4 \times 5 = \boxed{20}$ possible values

Problem 2: Prove that $2^n + 6 \cdot 9^n$ is always divisible by 7 for $\{n \in \mathbb{Z}^+\}$

$6 \equiv -1 \bmod 7$

$2 \equiv 9 \bmod 7$

$2^n \equiv 9^n \bmod 7$

$6 \cdot 9^n \equiv -1 \cdot 2^n = -2^n$

$2^n + 6 \cdot 9^n \equiv 2^n - 2^n \equiv 0 \bmod 7$

Proved

Homework

Problem 3: Mrs. Walter gave an exam in a mathematics class

of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walter entered?

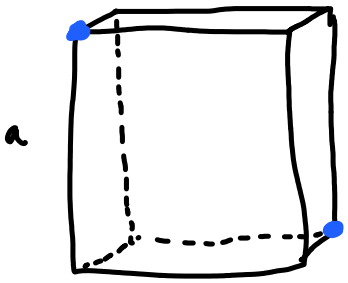
Problem 4: Which digits must we substitute for a and b in 30a0b03 so that the resulting integer is divisible by 13?

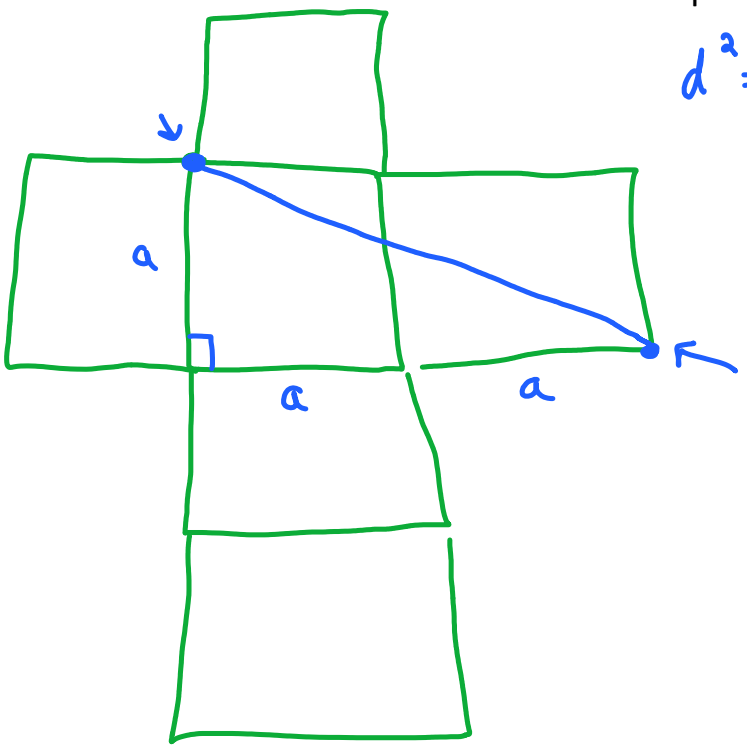
Problem 5: For how many positive integral values of $x \leq 100$ is $3^x - x^2$ divisible by 5?

Maths Circle Senior - More Triangle Inequality (Dec 1

Sunday, December 15, 2024 9:21 AM

Problem 1: A fly sits on one vertex of a wooden cube. What is the shortest path it can follow to the opposite vertex?





$d^2 = a^2 + (2a)^2$

$= a^2 + 4a^2$

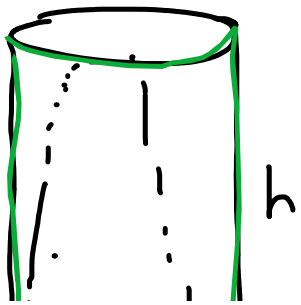
$= 5a^2$

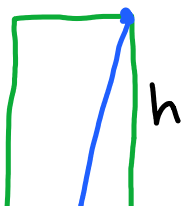
$d = \sqrt{5}\,a$

Problem 2: A fly sits on the outside surface of a cylindrical drinking glass. It must crawl to another point, situated on the inside surface of the glass. Find the shortest path possible (neglecting the thickness of the glass).

$2(\sqrt{h^2+r^2})$

$\sqrt{h^2+4r^2} + h$





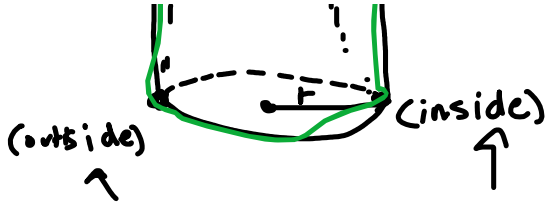
$d^2 = (2h$

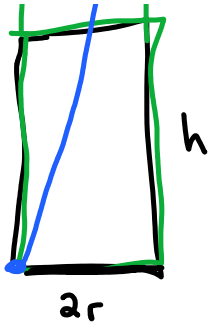
$= 4h^2$

5)

$)^2 + (2r)^2$

$+ 4r^2$

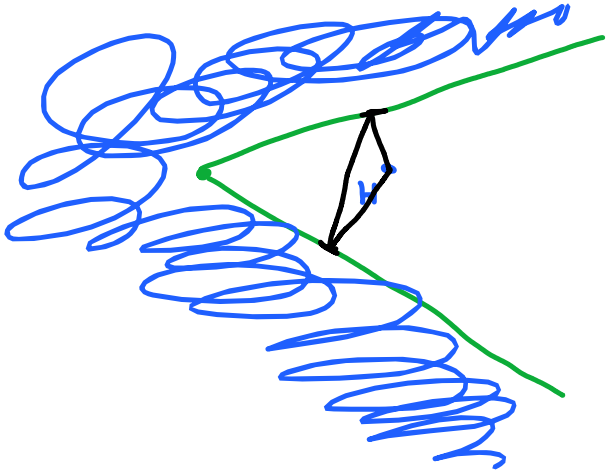




$= 4($

$d = 2\sqrt{}$

Problem 3: A woodsman's hut is in the interior of a peninsula which has the form of an acute angle. The woodsman must leave his hut, walk to one shore of the peninsula, then to the other shore, then return home. How should he choose the shortest such path?

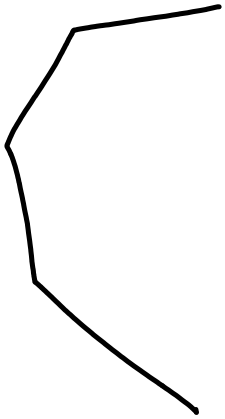


Problem 4: Prove that a convex polygon cannot have three sides, each of which is

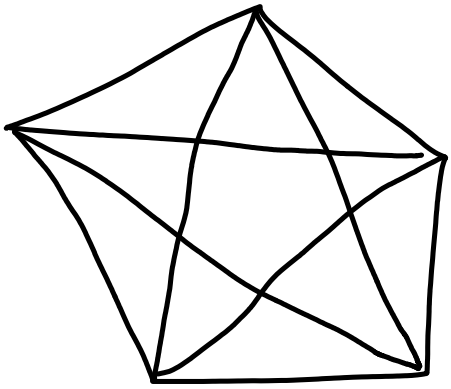
$h + r$)

$h^2 + r^2$

Problem 4: Prove that a convex polygon cannot have three sides, each of which is greater than the longest diagonal.



Problem 5: Prove that a convex pentagon (that is, a pentagon whose diagonals all lie inside the figure) has three diagonals which can form a triangle.



Problem 6: Two villages lie on opposite sides of a river whose banks are parallel lines. A bridge is to be built over the river, perpendicular to the banks. Where should the bridge be built so that the path from one village to the other is as short as possible?

Problem 7: Prove that the perimeter of a triangle is not greater than 4/3 the sum of its medians. (centroid divides a median in the ratio 2:1)

Maths Circle Senior - Number Theory

Sunday, August 3, 2025 9:32 AM

Problem 1:
How many integers between 1 and 1000 are divisible by 5 but not divisible by 7?

Divisible by 5: $\frac{1000}{5} = 200$ numbers

Divisible by $\underbrace{5 \text{ and } 7}_{35}$: $\left\lfloor \frac{1000}{35} \right\rfloor = 28$ numbers

$200 - 28 = \boxed{172}$

Problem 2:
Find the smallest positive integer n such that 13n ≡ 1 mod 100.

$\boxed{70}$

Problem 3:
Let S be the sum of all positive integers less than 100 that are relatively prime to ~~100. Find S.~~

47, 49, 51, 53

$40 \times 50 = \boxed{2000}$

Problem 4:
How many integers 1 ≤ n ≤ 1000 satisfy that n^2 ≡ 1 mod 8?

$n^2 \equiv 1 \bmod 8$

$2k+1$

n^2

$\boxed{500}$

$= 4k^2+4k+1$

$4k^2 - 1 =$

$= 4k(k+1) + 1 = n^2$

$(2k+1)(2k-1)$

even even

$4k(k+1) = n^2 - 1$

4 2

odd × odd

$= \text{odd} \times 8$

$4 \times 2 = 8$

$n^2 \equiv 1 \bmod 8$

Problem 5:
Let p and q be primes such that p + q = 100. How many such (unordered) pairs (p, q) exist?

Problem 6:
Let N be a positive integer such that N mod 3 = 2, N mod 4 = 3, and N mod 5 = 4. What is the smallest such N?

Problem 7:
A number has the property that when written in binary and reversed, its value doubles. What is the smallest such number?

Problem 8:
Find the number of positive integers less than 1000 whose base-8 representation ends with the digit 5.

Problem 9:
Let A be the 3-digit hexadecimal number “1F4”. Convert A to base-10 and then to binary.

Problem 10:
Let N be a positive integer such that its binary representation is 11011. Find N in base 10 and base 8.

Maths Circle Senior - Number Theory

Sunday, January 19, 2025 9:22 AM

Problem 1: Prove that there exist two powers of two which differ by a multiple of 1987.

$2^{1984} \equiv 2 \bmod 1987$

$2^{m} \equiv 2^{n} \bmod 1987$

$2^{1986} \equiv 1 \bmod 1987$

WLOG $m > n$

$m - n = 1986$

$2^{m-n} \equiv 1 \bmod 1987$

proved ✓

Problem 2: Prove that every integer > 6 can be represented as a sum of two integers > 1 which are relatively prime.

odd : for any odd number n,

$$(n-2) + 2 = n$$

even : for any even number n,

if n is twice of an odd number:

$$\left(\frac{n}{2}+2\right) + \left(\frac{n}{2}-2\right) = n \quad \checkmark$$

if n is twice of an even number.

$$\left(\frac{n}{2}+1\right)+\left(\frac{n}{2}-1\right)=n$$

Problem 3: Prove that if a and b are different integers, then there exist infinitely many positive integers n such that a + n and b + n are relatively prime.

$$\gcd(a+n, b+n)=1$$

$$\gcd(x,y)=\gcd(x-y,y)$$

$$\gcd(a-b, b+n)=1 \quad \rightarrow d$$

$$\gcd(d, b+n)=1$$

$$d=d_1^{p_1} d_2^{p_2} d_3^{p_3} \cdots d_k^{e_k}$$

Problem 4: Prove that for every integer k the numbers 2k+l and 9k+4 are relatively prime, and for numbers 2k-l and' 9k+4 find their greatest common divisor as a function of k.

Problem 5: Prove that for every positive integer s there exists a positive integer n with the sum of digits (in decimal system) equal to s which is divisible by s.

Maths Circle Senior - Parity

Saturday, May 3, 2025 9:47 PM

Problem 1: A 7×7 grid has numbers 1 through 49 placed in each square. Every time a number is even, it is painted red. What is the minimum number of rows that must contain an even number?

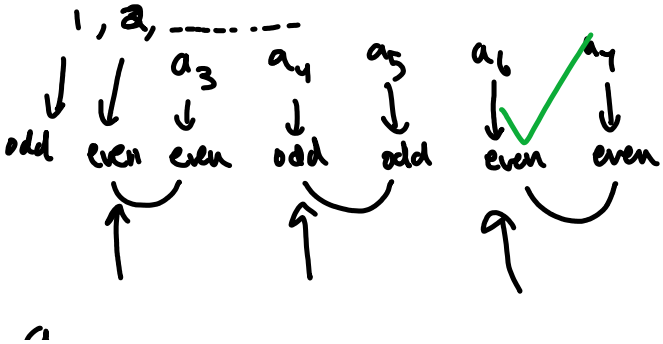
even numbers → 24

1 row → 7 numbers

$7 \times \boxed{4} = 28 > 24$

Problem 2: A sequence is defined by $a_1 = 1$, and for $n \geq 1$, $a_{n+1} = a_n + n$. What is the parity of a_{100}?

parity → even or odd?



$2(2k+1)$ = even

$a_{2(2k)}$ = odd

$100 = 2(50)$ → odd

↳ even

Problem 3: Let n be the smallest positive integer such that $1^2 + 2^2 + 3^2 + ... + n^2$ is divisible by 100. What is the parity of n?

Problem 4: You have 13 coins, each showing either heads or tails. You are told that at least one coin is heads. Every minute, you flip all coins showing heads and exactly one tails coin of your choice. Can all coins eventually show tails?

no

Problem 5: Let n be a positive integer such that when n is divided by 4, the remainder is 1, and when n is divided by 6, the remainder is 3. What is the parity of the least such n?

odd

Problem 6: A function f is defined on the integers such that f(n) = f(n−1) + f(n−2), with f(1) = 1 and f(2) = 3. What is the parity of f(2024)?

$n \equiv 1,2 \bmod 3 \rightarrow f(n) = \text{odd}$

$n \equiv 0 \bmod 3 \rightarrow f(n) = \text{even}$

$2024 \equiv 2 \bmod 3 \Rightarrow \text{odd}$

f(3) = 4 → even

f(4) = 7 → odd

f(5) = 11 → odd

f(6) = 18 → even

f(7) = 29 → odd

f(8) = odd

f(9) = even

Problem 7: Let n be the smallest positive integer such that n leaves a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, ..., and a remainder of 9 when divided by 10. What is the parity of n?

Problem 8: A 10×10 grid has 100 lights, all initially off. You press every button in row 1, then every button in row 2, and so on, until row 10. Each button toggles the light in its cell and its four neighbors (up, down, left, right). After all presses, how many lights remain on?

Problem 9: Let a and b be integers such that $a^2 \equiv b^2 \bmod 2^o$. What is the parity of $a - b$?

Problem 10: A circle has 2025 equally spaced points. How many distinct chords can be drawn connecting pairs of points such that the number of points between the endpoints is even?

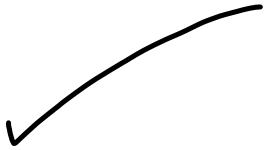
Maths Circle Senior - PHP

Saturday, June 14, 2025 11:07 PM

Problem 1:
In a group of 13 people, each person is assigned a birthday month (January to December). Prove that at least two people must share the same birthday month.

everyone (12) → different birthmonths

1 person → at least 1 month will have 2



Problem 2:
A drawer contains 10 black socks, 12 white socks, and 8 gray socks, all mixed together. What is the minimum number of socks you must pull out (in the dark) to guarantee that you have at least two socks of the same color?

3 socks → all different

4th one → 1 colour will have 2 socks

Problem 3:
You select 51 numbers from the set {1, 2, 3, ..., 100}. Prove that there exist two numbers such that one divides the other.

Fundamental Theorem of Arithmetic

$n = 2^k \cdot m$
↳ odd

- $k \geq 0$
- m is odd

Problem 4:
A list of 20 integers is made, each of which is from the set {1, 2, ..., 35}. Show that at least two numbers in the list differ by less than 2.

Problem 5:
In a room of 20 people, each person shakes hands with at least one other person. Prove that at least two people have shaken hands the same number of times.

Problem 6:
From the set {1, 2, 3, ..., 18}, select any 10 integers. Prove that two of the selected integers must add to 19.

Problem 7:
You randomly place 11 rooks on an 8×8 chessboard. Prove that at least two rooks must attack each other.

Problem 8:
A school has 200 students, and each student has selected one of the 12 zodiac signs. Prove that at least 17 students have the same sign.

Problem 9:
If 8 points are placed inside a square of side length 2, prove that at least two points are within √2 units of each other.

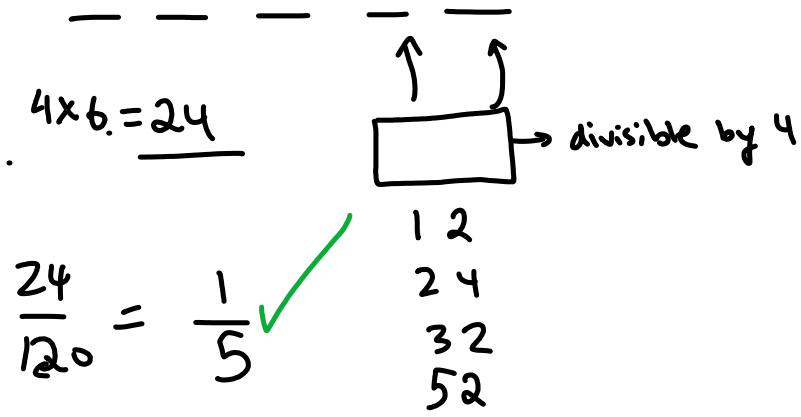
Problem 10:
Each of the numbers from 1 to 100 is colored either red, blue, or green. Prove that there exists two numbers of the same color such that one divides the other.

Maths Circle Senior - Probability

Saturday, April 19, 2025 11:14 PM

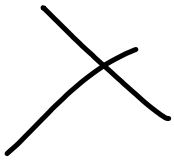
Problem 1: A 5-digit number is formed by randomly arranging the digits 1 through 5. What is the probability that the number is divisible by 4?

no repetition



Problem 2: A fair coin is tossed repeatedly until either two heads in a row or two tails in a row appear. What is the probability that this process ends with two heads?

k times tossed



Markov chain

Problem 3: Two positive integers less than or equal to 100 are chosen at random. What is the probability that they are relatively prime?

Problem 4: A fair 6-sided die is rolled repeatedly until the sum of the rolls is greater than or equal to 12. What is the probability that exactly three rolls are needed?

Problem 5: Two distinguishable standard 6-sided dice are rolled. What is the probability that the product of the numbers showing is a multiple of 6?

Problem 6: Let S be the set of all 3-digit numbers with distinct digits. What is the probability that a randomly chosen number from S is divisible by 5?

Problem 7: A bag contains 4 red balls and 6 green balls. Two balls are drawn at random without replacement. What is the probability that the two balls are of different colors?

Problem 8: Alice and Bob each flip a fair coin 5 times. What is the probability that they both get the same number of heads?

Problem 9: A drawer contains 3 pairs of indistinguishable socks: 1 red, 1 blue, and 1 green. If two socks are drawn at random without replacement, what is the probability they form a matching pair?

Problem 10: Three cards are drawn at random without replacement from a standard deck of 52 cards. What is the probability that all three cards are of the same suit?

Maths Circle Senior - Trigonometry

Sunday, July 20, 2025 9:34 AM

Problem 1:
Prove that:
sin(x) * tan(x) = sin²(x) / cos(x)

Problem 2:
Simplify the expression:
(cos(x) + sin(x))² + (cos(x) - sin(x))²

Problem 3:
Prove that:
(1 - sin²(x)) * (1 + tan²(x)) = cos²(x) * sec²(x)

Problem 4:
Prove the identity:
1 / (1 - sin(x)) + 1 / (1 + sin(x)) = 2 * sec²(x)

Problem 5:
Simplify:
sin(x) * csc(x) + cos(x) * sec(x)

Problem 6:
Prove:
(tan(x) + cot(x))² = sec²(x) + csc²(x) + 2

Problem 7:
Simplify the expression:
(sec(x) - tan(x)) * (sec(x) + tan(x))

Problem 8:
Prove:
cos(x) / (1 - sin(x)) + cos(x) / (1 + sin(x)) = 2 * sec(x)

Problem 9:
Prove the identity:
1 + tan(x) = sin(x) / (cos(x) - sin(x) * tan(x))

Problem 10:
Simplify the expression:
(sec²(x) - 1)(csc²(x) - 1)

Maths Circle Senior - Trigonometry

Saturday, May 10, 2025 10:56 PM

Trigonometric Ratios:

$\sin\theta = \frac{p}{h}$

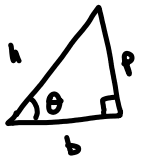
$\cos\theta = \frac{b}{h}$

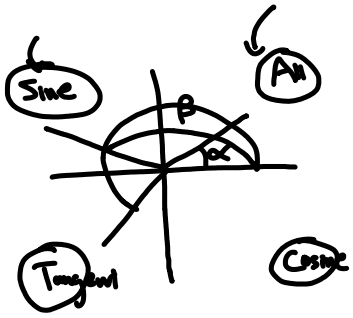
$\tan\theta = \frac{p}{b}$

$\text{cosec}\,\theta = \frac{h}{p}$

$\sec\theta = \frac{h}{b}$

$\cot\theta = \frac{b}{p}$





Trigonometric Identities:

$\sin^2\theta + \cos^2\theta = 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

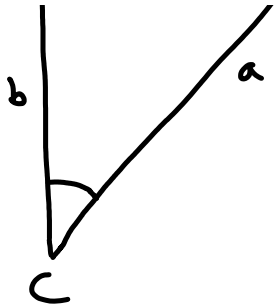
$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \text{"} \quad + \quad \text{"}$

Sine and Cosine Law:

sine law:

$\frac{a}{\quad} = \frac{b}{\quad} = \frac{c}{\quad}$

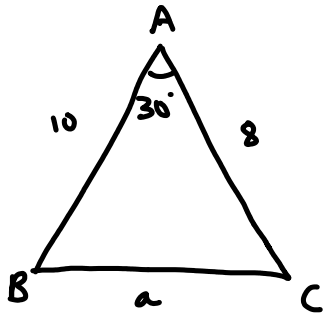


$$\frac{}{\sin A} = \frac{}{\sin B} = \frac{}{\sin C}$$

Cosine law:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Problem 1: In triangle ABC, angle A is 30 degrees, side b = 8, and side c = 10. Find the length of side a.



$$a = \sqrt{10^2 + 8^2 - 2(10)(8)(\cos 30)}$$

$$= \sqrt{164 - 2(10)(8)\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \sqrt{164 - 80\sqrt{3}}$$

Problem 2: Evaluate the exact value of sin(75°) using angle sum identities.

$\sin 75$

$= \sin(45+30)$

$= \sin(45)\cos(30) + \cos(45)\sin(30)$

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

Problem 3: Given that sin(x) + cos(x) = √2, find the exact value of sin(2x).

$(\sin x + \cos x)^2 = 2$

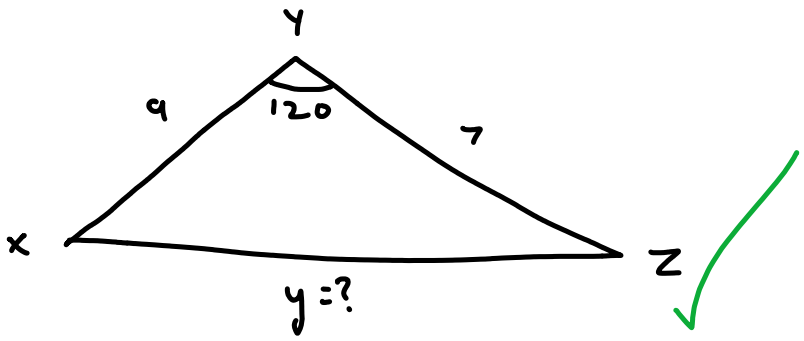
$\sin^2 x + \cos^2 x + 2\sin x \cos x = 2$

$1 + 2\sin x \cos x = 2$

$2\sin x \cos x = 1$

$\boxed{\sin 2x = 1}$

Problem 4: In triangle XYZ, angle Y is 120 degrees, side x = 7, and side z = 9. Use the Law of Cosines to find the length of side y.

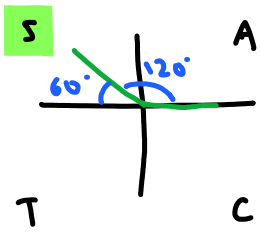


$y = \sqrt{9^2 + 7^2 - 2(7)(9)(\cos 120)}$

$\cos 120 = -\cos 60 = -\frac{1}{2}$

$y = \sqrt{81 + 49 + 63}$

$= \sqrt{193}$

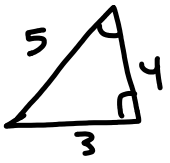


Problem 5: Find the general solution to the equation 2sin(x)cos(x) = 1.

$n\pi + \frac{\pi}{4}$

$180°n + 45°$

Problem 6: Let sin(A) = 3/5 with A in the first quadrant. Find the exact value of sin(2A) and cos(2A).



$\cos A = \frac{4}{5}$

$\sin 2A$

$= 2\sin A \cos A$

$= 2 \times \frac{3}{5} \times \frac{4}{5}$

$= \frac{24}{25}$

$\cos 2A$

$= 1 - 2\sin^2 A$

$= 1 - 2 \times \frac{9}{25}$

$= 1 - \frac{18}{25}$

$= \frac{7}{25}$

Problem 7: Solve for x in the interval $[0, 2\pi)$: $2\sin^2(x) - 3\sin(x) + 1 = 0$.

$0 \le x < 2\pi$

$0 \le x < 360$

let $\sin x = a$

$2a^2 - 3a + 1 = 0$

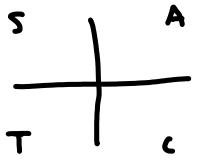
$a = 1$ or $a = \frac{1}{2}$

$\sin x = 1$ or $\sin x = \frac{1}{2}$

$x = 90$ or $x = 30, 150$

$\therefore x = 30, 90, 150$

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$



∠A	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cot A	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cosec A	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Problem 8: If tan(θ) = 3 and θ is in the second quadrant, find sin(2θ).

Problem 9: Prove the identity: $(1 - \sin^2 x)(1 + \tan^2 x) = \cos^2 x + \sin^2 x$.

Problem 10: In triangle PQR, sides p = 5, q = 7, and angle R = 60 degrees. Find the area of triangle PQR.

Maths Circles Junior - Miscellaneous

Saturday, August 16, 2025 9:37 AM

1) The sum of 3 integers is 1. Their product is 36. Find the largest of these 3 integers.

$\boxed{6}, -3, -2$

2) p and ptl are 2 primes. Then $\frac{p(ptl)}{2ptl}$ lies between

a) $\frac{4}{5}$ and 1

b) 1 and $\frac{6}{5}$

c) $\frac{6}{5}$ and $\frac{7}{5}$

d) $\frac{7}{5}$ and $\frac{8}{5}$

$\boxed{e}$ none of the above

$\boxed{\frac{6}{5}}$

3) For how many integer values of n is $4000 \cdot \left(\frac{2}{5}\right)^n$ an integer?

$4000 \cdot \left(\frac{2}{5}\right)^n = \left(2^5 \cdot 5^3\right) \cdot \left(2 \cdot 5^{-1}\right)^n$

$= 2^{n+5} \cdot 5^{3-n}$

$n+5 \geq 0$ $\quad 3-n \geq 0$

$n \geq -5$ $\quad n \leq 3$

$-5 \leq n \leq 3$

$3 - (-5) + 1 = \boxed{9}$

4) Ratio of length to width of rectangle is 4:3. Diagonal is d, and area is kd^2, find k.

5) 2 right circular cylinders with same volume, but 1 has 10% more radius. Find relationship between their heights.

$V = \pi r^2 h$

$\pi r^2 h$

$r_2 = \frac{11 r_1}{10}$

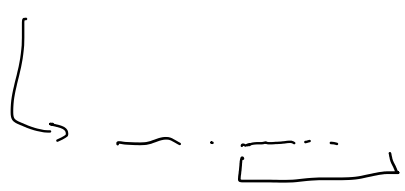
$\pi r_1^2 h_1 = \pi r_2^2 h_2$

$r_1^2 h_1 = \frac{121 r_1^2}{100} h_2$

$h_1 = \frac{121}{100} h_2$

↳ 21% greater

6) minimum # of digits to the right of the decimal point to express $\left(\frac{123456789}{2^{26} \cdot 5^4}\right)$ as a decimal



$= \frac{123456789}{2^{22} \cdot 2^4 \cdot 5^4}$

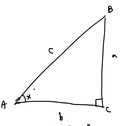
$= \frac{123456789}{2^{22} \cdot 10^4}$

$= \frac{12345.6789}{2^{22}}$

Maths Circles Junior - Trigonometry

Friday, July 18, 2025 6:14 PM

$1-\cos^2\theta = \sin^2\theta$



sin → sine
cos → cosine
tan → tangent
cosec → cosecant
sec → secant
cot → cotangent

$a^2+b^2=c^2$

$\sin x = \frac{a}{c}$ $\quad \cos x = \frac{b}{c}$ $\quad \tan x = \frac{a}{b}$

$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{c}{a}$ $\quad \sec x = \frac{1}{\cos x} = \frac{c}{b}$ $\quad \cot x = \frac{b}{a}$

soh cah toa

sin = opposite / hypotenuse

cos = adjacent / hypotenuse

$\sin^2\theta+\cos^2\theta = 1$

$1+\tan^2\theta = \sec^2\theta$

$1+\cot^2\theta = \operatorname{cosec}^2\theta$

Problem 1

Prove that:

$\sin^2(\theta) + \cos^2(\theta) = 1$

$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$

$= \frac{a^2+b^2}{c^2}$ ✓

$= \frac{c^2}{c^2}$

$= 1$

Problem 2

Prove that:

$1 + \tan^2(\theta) = \sec^2(\theta)$

$= 1 + \frac{a^2}{b^2}$

$= \frac{b^2}{b^2} + \frac{a^2}{b^2}$

$= \frac{a^2+b^2}{b^2}$ ✓

$= \frac{c^2}{b^2}$

$= \sec^2\theta$ □

Problem 3

Prove that:

$1 + \cot^2(\theta) = \csc^2(\theta)$

$= 1 + \frac{b^2}{a^2}$

$= \frac{a^2}{a^2} + \frac{b^2}{a^2}$ ✓

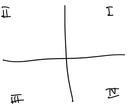
$= \frac{a^2+b^2}{a^2}$

$= \frac{c^2}{a^2}$

$= \csc^2\theta$ □

Problem 4

If $\tan(\theta) = 3/4$ and θ is in Quadrant I, find $\sin(\theta)$ and $\cos(\theta)$.



Problem 5

If $\sin(\theta) = 5/13$ and θ is in Quadrant I, find the value of $\cos(\theta)$.

Problem 6

Simplify the expression:

$[\sin(\theta) / (1 + \cos(\theta))] + [\sin(\theta) / (1 - \cos(\theta))]$

$\frac{\sin\theta}{1+\cos\theta} + \frac{\sin\theta}{1-\cos\theta}$

$= \sin\theta \left[\frac{1-\cos\theta + 1+\cos\theta}{(1+\cos\theta)(1-\cos\theta)} \right]$

$= \frac{\sin\theta\,(2)}{1-\cos^2\theta}$

$= \frac{2\sin\theta}{\sin^2\theta}$

$= \frac{2}{\sin\theta}$

Problem 7

If $\sin(\theta) = 3/5$ and θ is in Quadrant I, find the value of $\tan(\theta)$.

Problem 8

Simplify the expression:

$(1 - \cos^2(x)) / \sin(x)$

$\frac{1-\cos^2 x}{\sin x}$

$= \frac{\sin^2 x}{\sin x}$

$= \sin x$ ✓

Problem 9

Prove the identity:

$1 / (1 + \tan^2(x)) = \cos^2(x)$

$= \frac{1}{\sec^2 x}$

$= \cos^2 x$

(b) $\frac{1+\tan^2 x}{1+\cot^2 x} = \tan^2 x$

$= \frac{\sec^2 x}{\operatorname{cosec}^2 x}$

$= \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}$

$= \tan^2 x$

ii) $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

$= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)$

$= (\sin^2 x - \cos^2 x)$

$= 1 - \cos^2 x - \cos^2 x$

$= 1 - 2\cos^2 x$

$a^2 - b^2 = (a+b)(a-b)$

$a = \sin^2 x$

$b = \cos^2 x$

Maths Circle Senior - Sequences and Series

1) A sequence is defined by $a_1 = 2$, $a_{n+1} = a_n + 3$.

Find a_{50}.

$a + (n-1)d$

$= 2 + (n-1)3$

$= 2 + (50-1)3$

$= 2 + (49)3$

$= 2 + 147$

$= \boxed{149}$

2) Solve $\sum_{n=1}^{20} \frac{1}{n(n+1)}$ ($\sum$ (sigma) → summation)

telescoping

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \frac{1}{3} - \cdots + (\frac{1}{20} - \frac{1}{21})$

$= 1 - \frac{1}{21}$

$= \frac{20}{21}$

3) $a_1 = 7$, $a_{15} = 49$ of an A.P., find S_{15}

$S = \frac{15}{2}(7 + 49) = 15(28) = \boxed{420}$

4) The sequence b_n is defined by

$b_1 = 1$, $b_{n+1} = b_n + \frac{1}{b_n}$

modify it → approximation

Find the integer closest to b_{20}.

Maths Circle Senior – Probability

1) A fair 6-sided dice was rolled. What is the expected value of the square of the number rolled?

$$E(r) = \sum P(r) = \frac{1^2 + 2^2 + 3^2 + \dots + 6^2}{6}$$

$$= \frac{91}{6}$$

2) From a standard 52 card deck, if 1 card is drawn, what is the expected value of the number of the card?

suits → diamond, clover, spades, hearts

↓ 13 ↓ 13 ↓ 13 ↓ 13

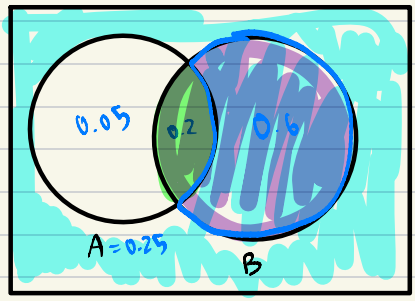
1, 2, 3, ... 10, 11, 12, 13

1, 2, 3, ..., 13

$$E(x) = \frac{1+2+\dots+13}{13} = \frac{13 \times 14}{2} \cdot \frac{1}{13} = \boxed{7}$$

3) Events A and B are independent with $P(A \cap B) = 0.2$ and $P(A' \cap B) = 0.6$.

a) Find $P(B)$



$0.6 + 0.2 = 0.8$

b) Find $P(A \cup B)$

$P(A \cap B) = P(A) \times P(B)$

$0.2 = P(A) \times 0.8$

$P(A) = 0.25$

$P(A \cup B) = 0.6 + 0.25 = 0.85$

Maths Circle Senior – Probability 2

1) Let A and B be independent events, where $P(A)=0.6$ and $P(B)=x$.

a) Write down an expression for $P(A \cap B)$ in terms of x

$P(A) \times P(B) = P(A \cap B) = 0.6x$

b) given that $P(A \cup B) = 0.8$

(i) Find x.

$0.8 = 0.6 + x - 0.6x$

$x = 0.5$

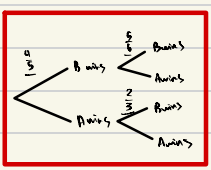
(ii) Find $P(A \cap B)$.

$P(A \cap B) = 0.3$

c) Explain why these events are NOT mutually exclusive.

$P(A \cap B) \neq 0$

2) Bill and Andrea play 2 games of tennis. The probability that Bill wins the first game is $\frac{4}{5}$. If Bill wins the first game, the probability that he wins the second game is $\frac{5}{6}$. If he loses the first game, the probability that he wins the second game is $\frac{2}{3}$.



C Diagram ↗

a) Find the probability that Bill wins the first game and Andrea wins the second.

$\frac{4}{5} \times \left(1 - \frac{5}{6}\right) = \frac{4}{5} \times \frac{1}{6_3} = \boxed{\frac{2}{15}}$

b) Find the probability that Bill wins at least one game.

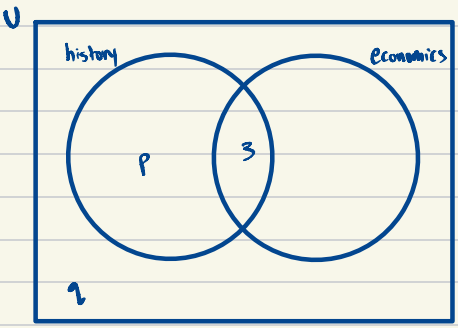
$\left(\frac{4}{5}\right)\left(\frac{5}{6}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{5}\right)\left(\frac{2}{3}\right) = \boxed{\frac{14}{15}}$

c) Given that he has won at least 1 game, find the probability that he wins both.

$P(A|B) = \dfrac{P(B|A)P(A)}{P(B)} = \dfrac{\frac{2}{5} \times \frac{5}{6_3}}{\frac{14}{15_7}} = \boxed{\frac{5}{7}}$

3)

In a group of 20 girls, 13 take history, 8 take economics, and 3 take both.



a) (i) Find p.

$p + 3 = 13$

$p = 10$

(ii) Find q.

$p + q + 3 + 5 = 20$

$q = 2$

b) A girl is selected at random. Find the probability that she takes economics but not history.

$$1 - \frac{15}{20} = \frac{5}{20} = \frac{1}{4}$$

Maths Circle Senior — Miscellaneous

1) remainder when $2^{202}+202$ is divided by $2^{101}+2^{51}+1$?

Hint: $\dfrac{2^{202}+202}{2^{101}+2^{51}+1}$

$= \dfrac{2^2 \cdot 2^{200}+202}{2\cdot 2^{100}+2\cdot 2^{50}+1}$

Let $x = 2^{50}$

$= \dfrac{4x^4+202}{2x^2+2x+1}$

$2x^2+2x+1 \overline{)\,4x^4+202}$

$2x^3-2x+1$

$-4x^4-4x^3-2x^2$

$-4x^3-2x^2$

$+4x^3+4x^2+2x$

$2x^2+2x+202$

$-2x^2-2x-1$

201

2) What are the last 2 digits of 7^{2018}?

49

3) If $y+4=(x-2)^2$, $x+4=(y-2)^2$, $x\neq y$, $x^2+y^2=?$

Hint: First subtract equations, then add them

sub $x^2-y^2-4x+4y=y-x$

$(x^2-y^2)=3(x-y)$

$(x+y)(x-y)=3(x-y)$

$x+y=3$

add $x^2+y^2-4x-4y+8=x+y+8$

$x^2+y^2=5(x+y)$

$=5(3)$

$=15$

Maths Circle Senior – Geometry

1) A circle with center O and radius 5 is tangent to both axes in the first quadrant. Find the equation of the circle.

$$(x-5)^2 + (y-5)^2 = 25$$

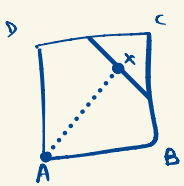
2) The points A(0,0) B(6,0) and C(x,4) form an isosceles triangle. Find all values of x.

$AC = BC$ or $AB = AC$

$$AC^2 = x^2 + 4^2$$
$$BC^2 = (x-6)^2 + 4^2$$
$$x^2 = (x-6)^2$$
$$\boxed{x = 3}$$

$$x^2 + 4^2 = 8^2$$
$$x = \sqrt{20}$$ ← simplify radicals
$$\boxed{\pm 2\sqrt{5}}$$

3) Square has vertices A(0,0), B(a,0) C(a,a), D(0,a).
Find the distance from A to the line passing through the midpoints of BC and CD.



$M\left(a, \frac{a}{2}\right)$ $N\left(\frac{a}{2}, a\right)$

$$m_{MN} = \frac{a - \frac{a}{2}}{\frac{a}{2} - a} = -1$$

$$y - \frac{a}{2} = -1\left(x - a\right)$$
$$y = -x + \frac{3a}{2}$$

$$\frac{\left|0 + 0 - \frac{3a}{2}\right|}{\sqrt{(-1)^2 + 1^2}} = \frac{\frac{3a}{2}}{\sqrt{2}} = \frac{3a}{2\sqrt{2}}$$

closest distance from point to line

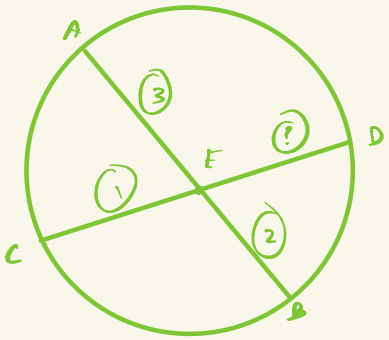
$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$ax + by + c = 0$ $\quad (x_1, y_1)$

4) Chords AB and CD of a circle intersect at point E inside the circle. Find ED if AE = 3, CE = 1, EB = 2.

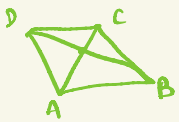
$ED \times 1 = 3 \times 2$

$ED = 6$



5) Points A(1,2) B(5,3) and C(2,7) are vertices for parallelogram ABCD. Find coordinates of D and area of parallelogram.

$D(x_D, y_D)$



$\left(\frac{1+2}{2}, \frac{2+7}{2}\right) = \left(\frac{3}{2}, \frac{9}{2}\right)$

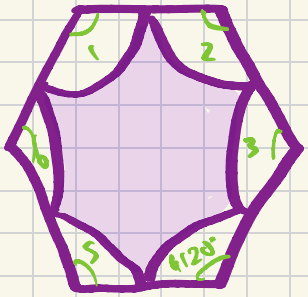
$\left(\frac{x_D+5}{2}, \frac{y_D+3}{2}\right) = \left(\frac{3}{2}, \frac{9}{2}\right)$

$(x_D, y_D) = (-2, 6)$

Area = 19

Maths Circle Senior – Geometry

1)



If the side length of hexagon is 6, then find the area of the shaded region.

Formula for angle for (regular) nth side polygon: $(n-2)(180)$

$\frac{1}{3} \times 6 = 2$ circles, $r = 3$

area of circles $= \pi \cdot 3^2 \times 2 = 18\pi$ → -18π

total area of hexagon ←

$\frac{3\sqrt{3}(3\times 6)}{2} = 54\sqrt{3}$ ←

$\therefore$ area of shaded region $= 54\sqrt{3} - 18\pi$

2) Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of AB and E be the midpoint of AC. The angle bisector of $\angle BAC$ intersects DE and BC at F and G, respectively. What is the area of FDBG?

Maths Circles Senior - Coordinate Geometry

1) Find distance between $(7,-2)$ and $3x-4y=20$.
(x,y) $Ax+By+C=0$

$$d = \frac{|A x_1 + B y_1 + C|}{\sqrt{A^2+B^2}}$$

$$= \boxed{\frac{9}{5}}$$

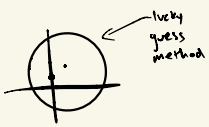
2) Find the area of a triangle with vertices $(0,0), (4,0), (2,6)$.
(x_1,y_1) (x_2,y_2) (x_3,y_3)

$$\text{Area} = \frac{1}{2}\left| x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2) \right|$$

$$= \frac{1}{2}(24)$$

$$= \boxed{12}$$

3) Circle with center $(2,3)$ and radius 5 is tangent to a line $y=mx+1$.
Find all positive values of m.



$mx-y+1=0$

$$d = \frac{|m(2)-7+1|}{\sqrt{m^2+1}}$$

$$= \frac{|2m-2|}{\sqrt{m^2+1}} = 5$$

$$21m^2+8m+21=0$$

$$D = 8^2 - 4(21)(21) < 0$$

$\boxed{\text{no real solutions}}$

4) Find coordinates of the reflection of $(3,4)$ across $y=x$.

$(4,3)$

5) Find the number of integer lattice points in or on the circle $x^2+y^2=25$

$4(0+4)+1 = \boxed{33}$

6) Find points of intersection between $x^2+y^2=25$ and $y=2x+1$

*

Maths Circle Senior - Combinatorics

1) Two fair 6-sided dice are rolled. What is the probability that the product is divisible by 6 but not by 9?

2 & 3

Case 1: one die divisible by 6

6+6-1 = 11 outcomes ←

Case 2: One die {2,4}, {3}

4

11+4 = 15

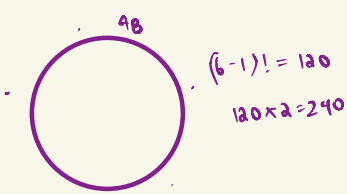
{3,6}, {6,3}, {3,3}, {6,6} → 4

15-4 = 11

Probability = $\frac{11}{36}$

2) 7 people sit at a circular table. 2 of them (A&B) refuse to sit next to each other. How many distinct seatings are there? (rotations are identical, reflections are distinct)

circle: (7-1)! = 720



(6-1)! = 120

120×2 = 240

720 - 240 = 480

3) From 10 men and 9 women, a committee of 6 is chosen. How many committees contain at least 3 women without having all members being a woman?

Case 1: 3M & 3W

$\binom{9}{3}\binom{10}{3} = 10080$

Case 2: 4W & 2M

$\binom{9}{4}\binom{10}{2} = 5670$

Case 3: 5W & 1M

$\binom{9}{5}\binom{10}{1} = 1260$

$10080 + 5670 + 1260 = \boxed{17010}$

4) What is the coefficient of x^{10} in the expansion of $(x^3+x^2+x+1)^8$?

[6728]

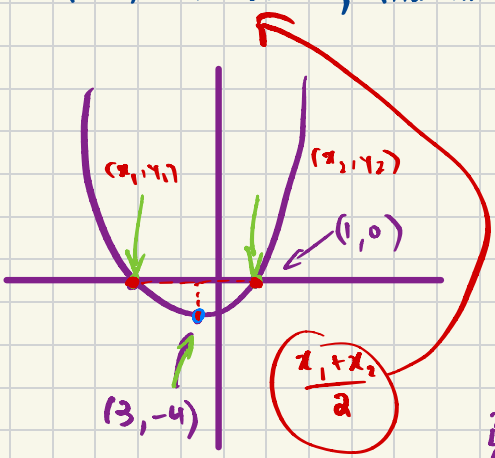
Maths Circle Junior - Miscellaneous

1) Simplify $\dfrac{(2^{x+2})(4^{x-1})}{8^x}$

$4 = 2^2$, $8 = 2^3$

$$\frac{(2^{x+2})(2^{2x-2})}{2^{3x}} = 2^{x+2+2x-2-3x} = 2^0 = 1$$ ✓

2) If $f(x) = x^2 - 6x + 5$, find the minimum value of $f(x)$.



$f(x) = a(x-h)^2 + k$

minimum $= (h, k)$

$x^2 - 6x + 5$

$a^2 \quad -2ab$

$b = 3$, $b^2 = 9$

$= x^2 - 6x + 9 - 4$

$= (x-3)^2 - 4$

$a(x-h)^2 + k$

$a = 1$, $h = 3$, $k = -4$

minimum value =

3) Find all integers n such that n^2+3n+2 is divisible by 5.

n^2+3n+2

$n=0$
$n=1$
2
3 ✓
4 ✓
5
6
7
8 ✓
9 ✓
10
11
12
13 ✓
14 ✓

$+5$ $+5$ $+5$ $+5$

$a = n_1$ / first term
d = common difference

$3, 8, 13, \ldots = 5n-2, \quad n \in \mathbb{Z}^+$ ✓
$4, 9, 14, \ldots = 5n-1, \quad n \in \mathbb{Z}^+$

belongs to integers

$5 \quad 5 \quad 5$
$3, 8, 13, 18, \ldots\ldots$

$a + (n-1)d$

$\boxed{a}, a+d, a+2d, a+3d, \ldots\ldots, a+(n-1)d$
1 2 3 n^{th} term

$a + (n-1)d$
$3 + (n-1)(5)$
$= 3 + 5n - 5$
$= 5n - 2$

4) A line passes through points $A(1,2)$ and $B(5,10)$. What is the equation of the line?

$b = 0$
$m = 2$

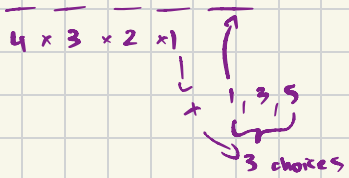
$y = 2x$ ✓

equation → $0 = 0$
$y = 2x$
expression → $2x$

5) How many different 5-digit numbers can be formed using digits $1, 2, 3, 4, 5$, if the number must be odd and the digits cannot repeat?

$4! \times 3$

$= 72$ ✓



Maths Circle Junior – Vieta's Formulae

Vieta's Formula for Quadratics:

$f(x) = ax^2 + bx + c = 0$, $x = p$ or $x = q$

roots → p & q

$$p + q = -\frac{b}{a}$$

$$pq = \frac{c}{a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sridhar Acharya Formula

1) $x^2 - 7x + 10 = 0$, roots are a and b, find $a^2 + b^2$.

without finding individual roots

$$a^2 + b^2 = (a+b)^2 - 2ab$$

$$= \left(-\frac{7}{1}\right)^2 - 2\left(\frac{10}{1}\right)$$

$$= 49 - 20$$

$$= \boxed{29}$$

2) If a and b are roots of $x^2 - 3x + 2 = 0$, find the quadratic whose roots are $\boxed{\frac{1}{a}}$ and $\boxed{\frac{1}{b}}$

$ax^2 + bx + c = 0$

sum: $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{3}{2}$

product: $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} = \frac{1}{2}$

$mx^2 + nx + 0$

$-\frac{n}{m}$ $\frac{0}{m}$

$$x^2 - \frac{3}{2}x + \frac{1}{2} = 0$$

$$2x^2 - 3x + 1 = 0$$

Vieta's Formula for Cubics:

$f(x) = ax^3 + bx^2 + cx + d = 0$, roots are x_1, x_2, x_3

1. $x_1 + x_2 + x_3 = -\frac{b}{a}$ → sum of the roots
2. $x_1x_2 + x_2x_3 + x_1x_3 = \frac{c}{a}$ → sum of the products of the roots
3. $x_1x_2x_3 = -\frac{d}{a}$ → product of the roots

3) the roots of $x^3 - 6x^2 + 11x - 6 = 0$ are $a, b,$ and c.
Find $ab + bc + ac$.

(2) → $\frac{11}{1} = 11$

4) For $x^3 + 3x^2 + 3x - 1 = 0$, find $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ for roots a, b, c.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ac}{abc} = \frac{3}{1} = 3$$

5) The polynomial $1x^3 + px^2 + qx + r = 0$ has roots 1, 2, and 3.
Find the equation.

$1+2+3 = -p$, $p = -6$

$1\times2 + 2\times3 + 1\times3 = q$, $q = 11$

$1\times2\times3 = -r$, $r = -6$

$$x^3 - 6x^2 + 11x - 6 = 0$$

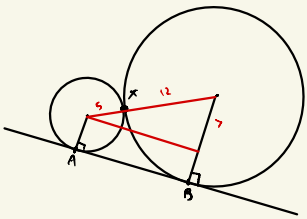
$$(x-1)(x-2)(x-3) = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

Maths Circle Junior – Geometry

tangent → meets at one point

1) Two circles with radii 5 and 12 are externally tangent. A common external tangent touches the circles at A and B respectively. Find AB.

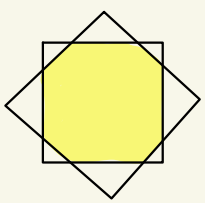


external tangent

$\sqrt{17^2 - 7^2} = 4\sqrt{15}$

$\sqrt{240} = \sqrt{16 \times 15} = 4\sqrt{15}$

2) Square of side length 10 is rotated about its center by 45°. What is the area of the overlapping region between the 2 squares?



"OCTAGON REGION"

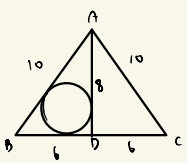
$100 - \left(4 \cdot \frac{1}{2}\left[10^2 - 50\right]\right)$

$= 200 - 100\sqrt{2}$

3) In $\triangle ABC$, $AB = AC = 10$, $BC = 12$. D is on BC such that AD is altitude. A circle is inscribed in $\triangle ABD$. Find its radius.

$\text{inradius} = \frac{\text{Area}}{\text{semiperimeter}}$ ← triangle that circle is inscribed in

3 ✗
1.5 ✗
$\frac{49}{\sqrt{3}}$ ✗
$\frac{3}{4}$ ✗

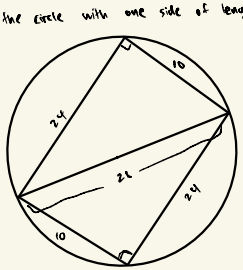


$\frac{10 + 8 + 6}{2} = 12$

$\frac{6 \times 8}{2} = 24$

$\text{inradius} = \frac{\text{area}}{\text{s-perimeter}} = \frac{24}{12} = 2$

4) A circle has radius 13. Find the maximum area of a parallelogram inscribed in the circle with one side of length 10.

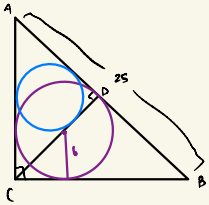


$\sqrt{26^2 - 10^2} = 24$

$\text{Area} = 24 \cdot 10 = 240$

5) In right triangle $\triangle ABC$ with $\angle C = 90^\circ$, a circle is inscribed with radius 6. Another circle is inscribed in the smaller right triangle formed by altitude $\overline{CD}$.

Find radius of smaller circle if $AB = 25$.



$r = 6$ $\quad a, b, c$

$c = 25$

$r = \frac{a+b-c}{2}$

$a+b = 37$

$a^2 + b^2 = 25^2$

$(a+b)^2 - (a^2+b^2) = 2ab$

$37^2 - 25^2 = 2ab$

$ab = 372$

$\frac{1}{2}ab = 186$

$r \cdot s = 6\left(\frac{a+b+c}{2}\right)$

$= 6 \cdot 31 = 186$

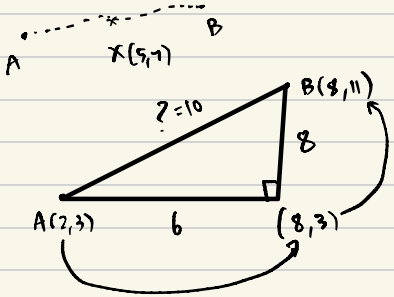
* Cauchy - schwarz inequality

Maths Circle Junior — Coordinate Geometry

$A(x_1, y_1)$ ..... $B(x_2, y_2)$

$X\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

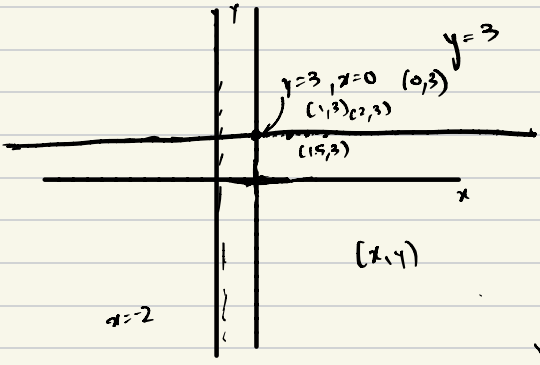
1) The points $A(2,3)$ and $B(8,11)$ are the ends of a diameter of a circle. Find the radius of that circle.

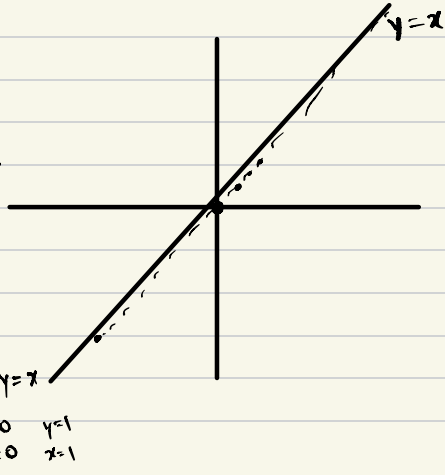


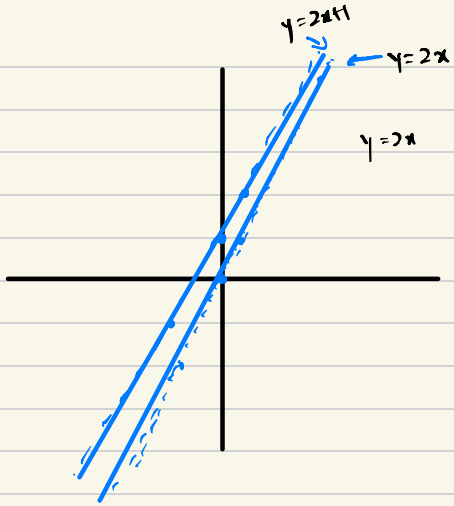
$A(x_1, y_1)$
$B(x_2, y_2)$

radius = 5

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$ → distance formula







$y = 2x$

$y=0$, $x=0$

$x=1$, $y=2$

$x=-1.5$, $y=-3$

1 ↓ 0 ↓

$y = 2x+1$

$x = 0$, $y=1$

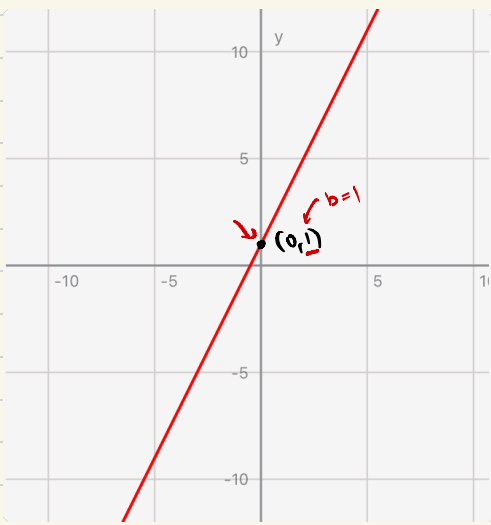
$x = 1$, $y=3$

$x = -1.5$, $y=-2$

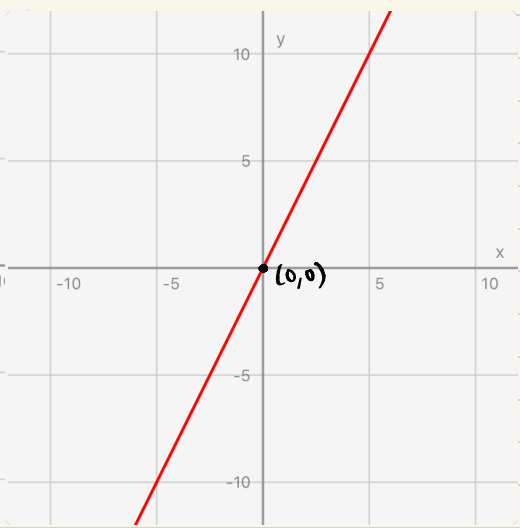
$y=mx+b$

$y=\underline{m}x+1$

$m=2$



$y = 2x+1$



$y = 2x$

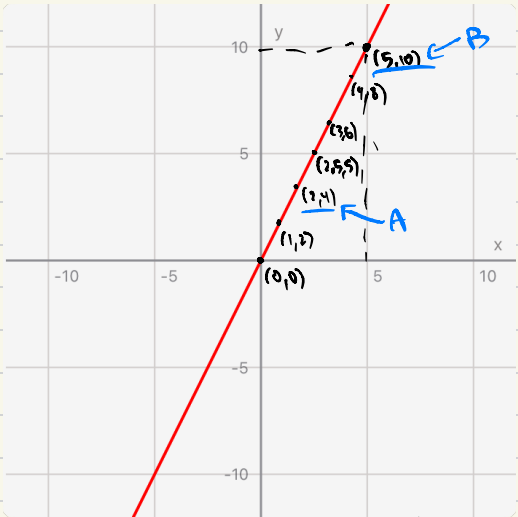
relation / equation : $y = mx + b$

(x, y) → coordinates of any point on the line

b = y-intercept

m = slope

$y = 2x$



$A(x_1, y_1)$ $B(x_2, y_2)$

$x_1 = 2$, $y_1 = 4$ $x_2 = 5$, $y_2 = 10$

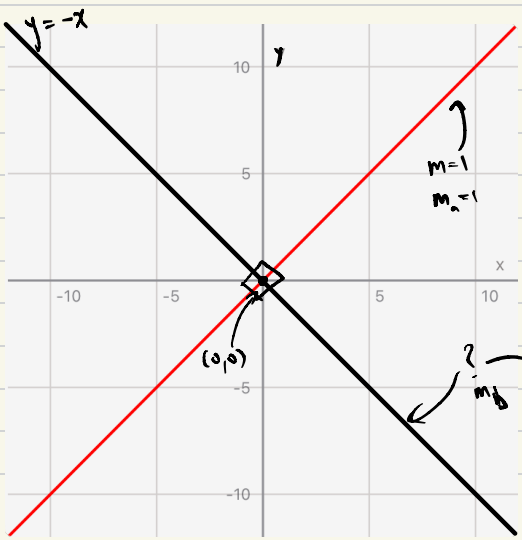
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{10 - 4}{5 - 2}$$

$$= \frac{6}{3}$$

$$= 2$$

$y = x$



$y = mx + b$

$b = 0$

$$m_2 = -\frac{1}{m_1}$$

$m_b = -1$

2) A line passing through the origin, has slope $= -\frac{3}{4}$. What is the equation of the line perpendicular to it and also passing through the origin.

($b_a = 0$; $m_a = -\frac{3}{4}$; $b_b = 0$)

$$y = -\frac{3}{4}x$$

$$m_b = -\left(\frac{1}{-\frac{3}{4}}\right)$$

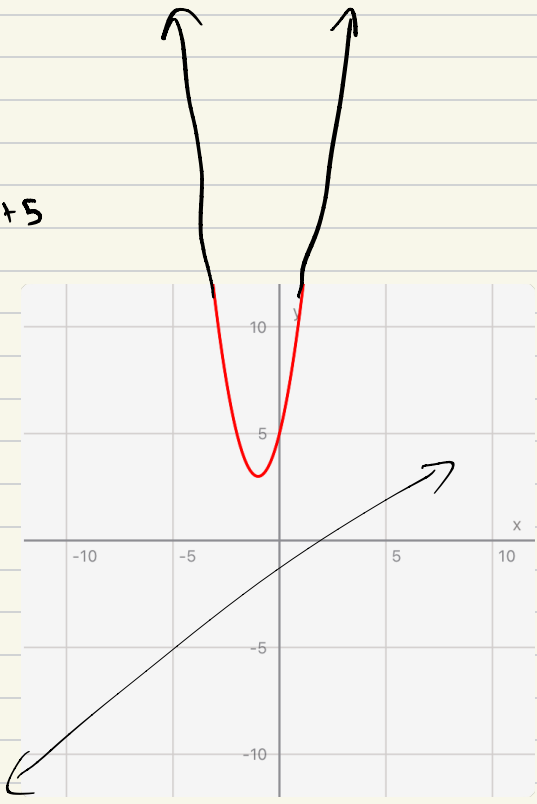
$$= \frac{4}{3}$$

$$y = \frac{4}{3}x$$

$y = 2x^2 + 4x + 5$

quadratic

parabola



$y = 5x^3 + 15x^2 + 2x + 1$ → cubic

Homework

1. A line passes through the points (0,0) and (2,4). What is its slope?

2. Find the length of the diagonal of a rectangle with corners (0,0),(0,6),(8,0), (8,6).

3. What is the equation of the line that passes through the origin and has slope 2?

more problems on the next page

1. What is the slope of a line perpendicular to the line with slope 2?

2. A line has equation y=3x+1. Find the slope of a line perpendicular to it.

3. The line through points (0,0) and (4,2) has slope \tfrac{1}{2}. What is the slope of a line perpendicular to it?

4. Find the equation of the line that passes through (2,3) and is perpendicular to y=2x+5.

5. What is the equation of the line perpendicular to y=-\tfrac{1}{3}x+4 that passes through the origin?

Maths Circle Junior – Cauchy-Schwarz Inequality

$$\left(\sum a_i b_i\right)^2 \le \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

$u_i = \frac{a_i}{\sqrt{b_i}}$, $v_i = \sqrt{b_i}$

1) Minimize $\frac{x^2}{y} + \frac{y^2}{x}$, $x+y=10$

$a_i \to x, y$

$b_i \to y, x$

$$\sum \frac{a_i^2}{b_i} \ge \frac{\left(\sum a_i\right)^2}{\sum b_i} \to \text{Titu's lemma}$$

$a_1 = x$, $b_1 = y$, $a_2 = y$, $b_2 = x$

$$\hookrightarrow \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2}$$

$$= \frac{x^2}{y} + \frac{y^2}{x} \ge \frac{(x+y)^2}{x+y}$$

$$\hookrightarrow \frac{x^3}{xy} + \frac{y^3}{xy} \ge \frac{(x^2+y^2)^2}{x^3y + y^2x}$$

$$\left(\sum a_i\right)^2 \le \left(\sum \frac{a_i^2}{b_i}\right)\left(\sum b_i\right)$$

$$\sum \frac{a_i^2}{b_i} \ge \frac{\left(\sum a_i\right)^2}{\sum b_i}$$

$$x^2y + y^2x = xy(x+y) = 10xy$$

$$\frac{x^2}{y} + \frac{y^2}{x} \ge \frac{(x^2+y^2)^2}{10xy} \to \frac{(100-2xy)^2}{10xy}$$

$$x^2+y^2 = (x+y)^2 - 2xy$$

$$= 100 - 2xy$$

$x+y=10$

$xy \leq 25$ $\quad x=5, y=5$

$$\frac{25}{5}+\frac{25}{5}=\boxed{10}$$

2) Find maximum of (using CS)

$$2x+y,\ x^2+y^2 \leq 5$$

CS: $(a_1^2+a_2^2)(b_1^2+b_2^2) \geq (a_1b_1+a_2b_2)^2$

$a_1=x,\ a_2=y,\ b_1=2,\ b_2=1$

$$(x^2+y^2)(2^2+1^2) \geq (2x+1\cdot y)^2$$

$$5(x^2+y^2) \geq (2x+y)^2$$

$$(2x+y)^2 \leq 5(5) \to 25$$

$$2x+y \leq 5$$

Maths Circle Junior - Combinatorics (2)

1) Two fair 6-sided dice are rolled. What is the probability that the product is divisible by 6 but not by 9?

Case 1: $\square \to 6$, $\square \to 1,2,3,4,5,6$

$6 + 6 - 1 = 11$

$3,6$
$6,3$
$3,3$
$6,6$ } 4

Case 2: $\square \to 2,4$, $\square \to 3$

$2 \cdot 2 = 4$

$11 + 4 = 15$

$15 - 4 = 11$

$\frac{11}{36}$

2) A box contains 4 red, 5 blue, and 6 green balls. Two are drawn without replacement. Given that at least one of the two is green, what is the probability that they are both green?

both green: $\frac{6^{3}}{15_{5}} \times \frac{5}{14_{7}} = \frac{1}{7}$

$P(G_1)$ = one of the two is green

$1 - P(G_1)$ = no balls are green $= 1 - \frac{9^{3}}{15_{5}} \times \frac{8^{4}}{14_{7}}$

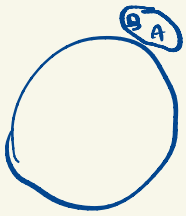
$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$

$= 1 - \frac{12}{35}$

$= \frac{23}{35}$

$\frac{\frac{1}{7}}{\frac{23}{35_{5}}} = \boxed{\frac{5}{23}}$

3) 7 people sit at a round table. Two of them refuse to sit next to each other. How many distinct seatings are possible?



$(n-1)!$

$(7-1)! = 720$

$2 \times (6-1)! = 2 \times 120 = 240$

$720 - 240 = 480$

A B C D E F G
G A B C D E F
F G A B . . . .

Maths Circle Junior – Combinatorics

1) How many different 4-digit numbers can be formed using the digits 1, 2, 3, 4 if repetition is <u>not</u> allowed?

$\_ \times \_ \times \_ \times \_ = 24$

$4 \times 3 \times 2 \times 1$

$4!$

2) How many ways can the letters of the word "LEVEL" be arranged?

E E V L L

2 L's
2 E's

$5! = 120$

$\frac{120}{2 \times 2} = 30$

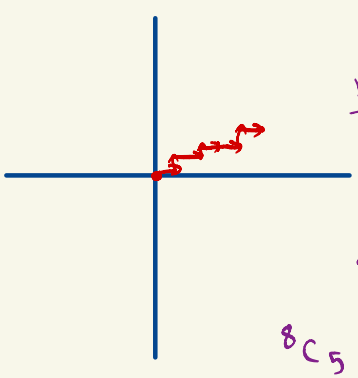
3) How many paths are there from (0,0) to (5,3) if only steps Right → and Up ↑ are allowed?

8 possible moves

5 Right , 3 Up

R U R U R R R U

← 5R & 3U



$^8C_5 = \frac{8!}{5!(8-5)!} = \frac{8!}{5!\,3!} = \frac{8 \times 7 \times 6 \times 5!}{5! \times 3!} = 56$

$^8C_3 = \frac{8!}{3!\,(8-3)!} = \frac{8!}{3!\,5!} =$

4) A bag contains 5R, 4B, & 3G. If 2 are drawn at random, what is the probability that they are the same?

R: $\frac{5}{12_3} \times \frac{4}{11} = \frac{5}{33}$

B: $\frac{4}{12_3} \times \frac{3}{11} = \frac{1}{11}$

G: $\frac{3}{12_4} \times \frac{2}{11} = \frac{1}{22}$

$\frac{5}{33} + \frac{1}{11} + \frac{1}{22} = \frac{10+6+3}{66}$

$= \frac{19}{66}$

5) How many distinct ways are there to seat 6 people around a circular table?

A	B	C	D	E	F
F	A	B	C	E	E

$(6-1)! = 120$

6) What is the probability that when 2 dice are rolled, the sum is divisible by 3?

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

12 multiples of 3

total = 36

$\frac{12}{36} = \boxed{\frac{1}{3}}$

Homework:

Two fair 6-sided dice are rolled. What is the probability that the product of the two numbers is divisible by 6?

From a group of 8 boys and 7 girls, how many different 5-person teams can be formed that contain **at least 2 girls**?

Four students sit in a row of 4 seats. Two friends insist on sitting **next to each other**. How many seatings are possible?

You flip a fair coin 6 times in a row. How many possible outcomes have **no two consecutive heads**?

Maths Circle Junior – Coordinate Geometry Part 2

1) The midpoint of a line segment is $(3,-2)$. If one of the endpoints is $(-7,1)$, find the other endpoint.

x: $\frac{x_2 + -7}{2} = 3$

$x_2 + 7 = 6$

$x_2 = -1$

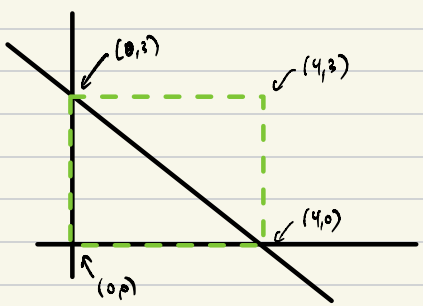
y: $\frac{y+1}{2} = -2$

$y+1 = -4$

$y = -5$

endpoint: $(-1,-5)$

2) A rectangle has 2 adjacent sides along $x=0$ and $y=0$. Its diagonal lies on the line $3x+4y=12$. Find rectangle area.



3) Line ℓ_1 has equation $y = 2x+3$.

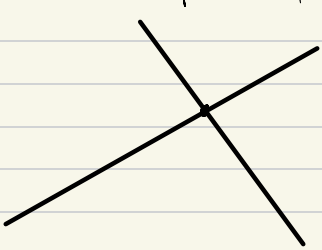
Line ℓ_2 is perpendicular to ℓ_1 and passes through $(4,1)$.

Find intersection point of ℓ_1 and ℓ_2.

$y = mx+b$

↳ (x,y)

$x=4, y=1$



$m_1 = 2$

$m_2 = -\frac{1}{2}$

ℓ_2: $y = mx+b$

$y = -\frac{1}{2}x + b$

$b = 3$

$y = -\frac{1}{2}x + 3$

$\ell_1 = \ell_2$

$2x+3 = -\frac{1}{2}x+3$

$x = 0$

$y = 3$

$\boxed{(0,3)}$