

**XXI GEOMETRICAL OLYMPIAD IN HONOUR OF  
I.F.SHARYGIN**  
**Final round. First day. 8 grade. Solutions**  
*July 31, 2025.*

1. (I.Kukharchuk, E.Galakhova.) A cyclic pentagon  $ABCDE$  is given. The diagonals  $AC$  and  $CE$  are equal and meet  $BD$  at points  $M$  and  $N$  respectively. It is known that  $BM = ND$ ,  $BC \neq CD$ . Prove that the reflection of  $C$  about the midpoint of  $BD$  lies on  $AE$ .

**Solution.** From the assumption we have  $CM \cdot MA = BM \cdot MD = DN \cdot BN = CN \cdot NE$ . Hence  $CM = CN$  or  $CM = EN$ . The first case is impossible because  $BC \neq CD$ , in the second case the midpoint of  $MN$  lies on the medial line of triangle  $ACE$  (fig. 8.1), which is equivalent to the required assertion.

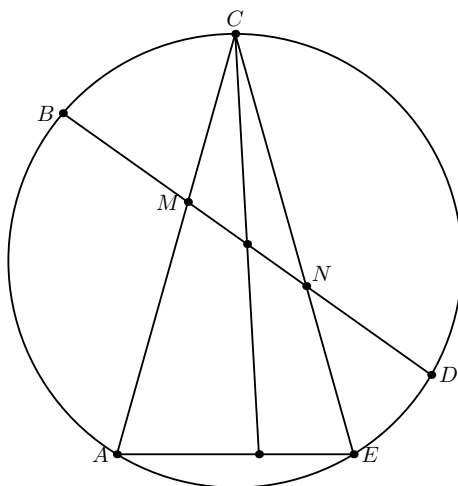


Fig. 8.1.

2. (L.Emelyanov) Let  $CH$  be an altitude of triangle  $ABC$ ; and  $CA'$ ,  $CB'$  be bisectors of triangles  $CBH$ ,  $CAH$  respectively. Prove that the circumcenter of triangle  $CA'B'$  coincides with the incenter of triangle  $ABC$  if and only if  $\angle ACB = 90^\circ$ .

**Solution.** Let the incenter  $I$  of triangle  $ABC$  coincide with the circumcenter of triangle  $A'B'C$ . Then it lies on the circumcircle of triangle  $A'BC$  as the common point of the bisector of angle  $B$  and the perpendicular bisector to  $A'C$ . Therefore  $\angle CIB = \angle CA'B$ . Similarly  $\angle CIA = \angle CB'A$  (fig. 8.2). Thus  $\angle AIB = 180^\circ - \angle A'CB'$ . On the other hand  $\angle AIB = 90^\circ + \angle A'CB'$ , which yields  $\angle C = 2\angle A'CB' = 90^\circ$ . Similarly we obtain the converse.

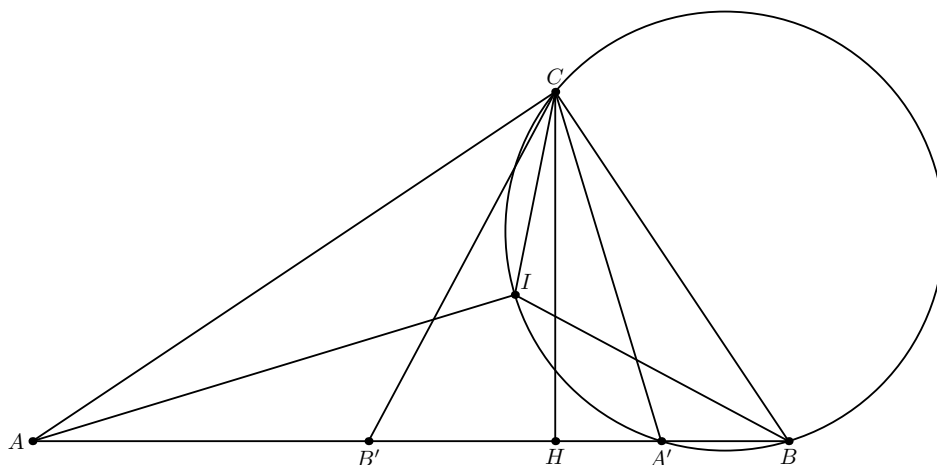


Fig. 8.2.

3. (F.Nilov) Can we choose more than six points on the plane not on a single line, and color them into three colors so that any line through two marked points of different colors contains exactly one more marked point, of the remaining color?

**Answer.** Yes, we can.

**Solution.** Let points  $A_1, B_1, C_1$  lie on a line  $\ell_1$ , points  $A_2, B_2, C_2$  lie on a line  $\ell_2$ , the lines  $A_1B_2$  and  $A_2B_1$  meet at point  $C_3$ , the lines  $A_1C_2$  and  $A_2C_1$  meet at point  $B_3$ , the lines  $B_1C_2$  and  $B_2C_1$  meet at point  $A_3$ . Then by the Pappus theorem  $A_3, B_3, C_3$  are collinear (fig. 8.3). Now coloring  $A_1, A_2, A_3$  into the first color,  $B_1, B_2, B_3$  into the second color, and  $C_1, C_2, C_3$  into the third one we obtain the required configuration.

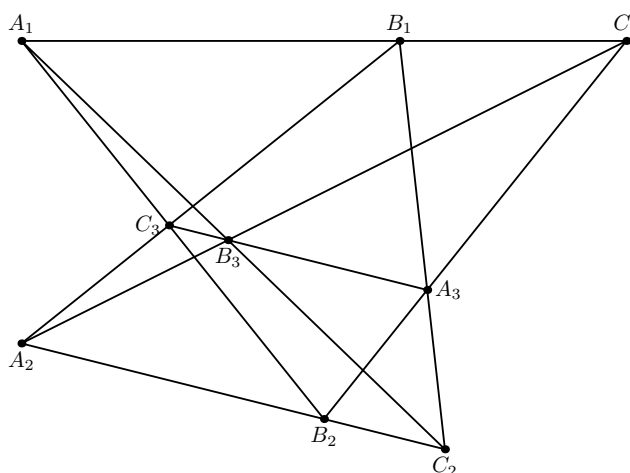


Fig. 8.3.

**Remark.** Using the addition of points on a cubic we can construct an example with an arbitrary great number of points: take a point  $P$  such that

$3nP = 0$  and color all points  $3kP$  into the first color, all points  $(3k + 1)P$  into the second one, and all points  $(3k - 1)P$  into the third color.

4. (L.Shatunov) Let  $AA_1$  and  $CC_1$  be bisectors of a triangle  $ABC$ , and  $B_0$  be the midpoint of the arc  $AC$  on the circumcircle of  $\triangle ABC$ , not containing  $B$ . The circumcircles of triangles  $AA_1B_0$  and  $CC_1B_0$  meet the lines  $BC$  and  $AB$  at points  $P$  and  $Q$  respectively. Prove that the incenter of  $\triangle ABC$  lies on  $PQ$ .

**First solution.** Let the line passing through  $I$  and parallel to  $AC$  meet  $BC$  at point  $P'$ . Then  $\angle P'IC = \angle ICA = \angle ICP'$ , therefore  $IP' = P'C$ . On the other hand  $IB_0 = B_0C$ , thus the triangles  $P'IB_0$  and  $P'CB_0$  are congruent, i.e.  $\angle B_0P'C = (180^\circ - \angle C)/2 = \angle IAB_0 = \angle CPB_0$ , and  $P'$  coincides with  $P$ . Similarly  $IQ \parallel AC$  (fig. 8.4).

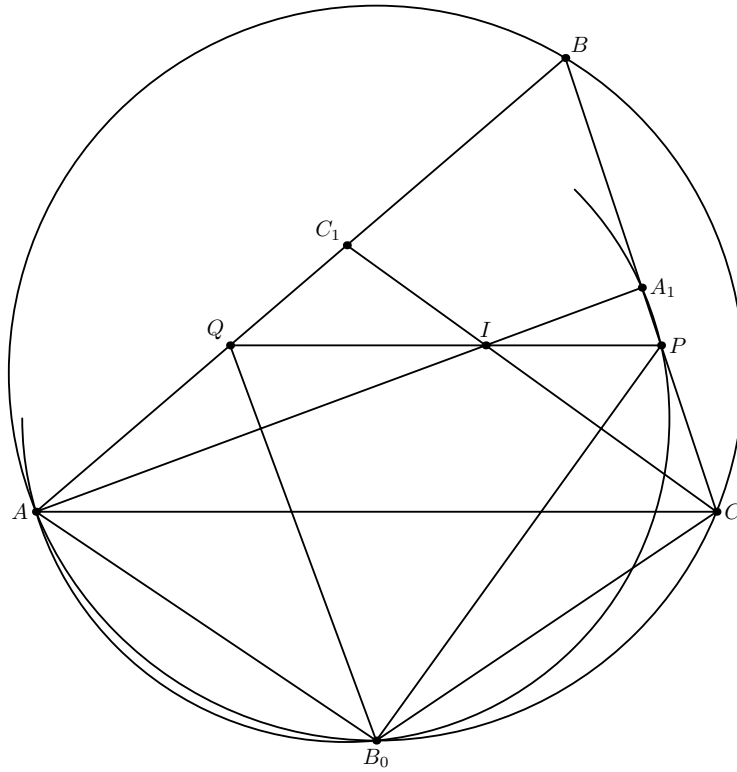


Fig. 8.4.

**Second solution.** Let  $A_0, C_0$  be the midpoints of arcs  $BC, AB$ ; and  $P'$  be the common point of  $A_0B_0$  and  $BC$ . Then  $AA_1P'B_0$  is a cyclic quadrilateral, because  $\angle AB_0A_0 = \angle BA_1A$ , thus  $P'$  coincides with  $P$ . Similarly  $Q$  lies on  $B_0C_0$ . Applying the Pascal theorem to the hexagon  $ABCC_0B_0A_0$  we obtain the required.

# XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

## Final round. Second day. 8 grade. Solutions

August 1, 2025.

5. (M.Volchkevich) The distance from the vertex of the right angle of a right-angled triangle to the bisector of its acute angle equals a quarter of its hypotenuse. Find all possible values of the angles of this triangle.

**Answer.**  $60^\circ$  and  $30^\circ$ , or  $36^\circ$  and  $54^\circ$ .

**Solution.** Let  $M$  be the midpoint of the hypotenuse  $AB$  of triangle  $ABC$ , and  $L$  be the reflection of  $C$  about the bisector of angle  $A$ . Then  $CL = AB/2 = CM$ , and two cases are possible.

1. Points  $M$  and  $L$  coincide. Then the bisector of angle  $A$  of triangle  $ACM$  coincides with the altitude of this triangle, therefore  $AC = AM = CM$ , and  $\angle A = 60^\circ$ .

2. Points  $M$  and  $L$  are different. Then  $AC = AL$ , and  $LC = CM = MA$ . Therefore  $\angle ALC = \angle LMC = 2\angle A$  (fig. 8.5). On the other hand  $2\angle ALC + \angle A = 180^\circ$ . Thus  $\angle A = 36^\circ$ .

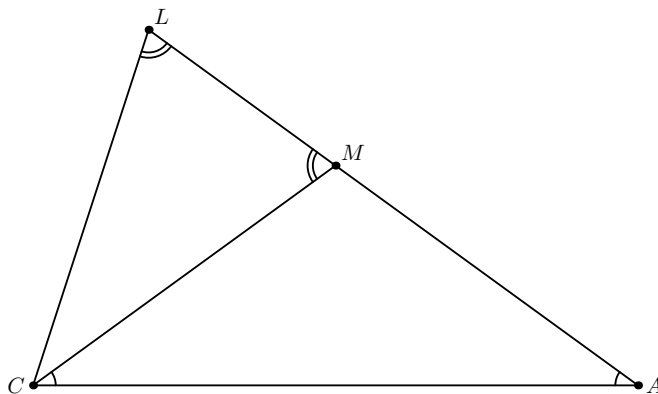


Fig. 8.5.

6. (I.Mikhailov) Let  $ABCD$  be a convex quadrilateral with  $\angle ABD = \angle ACD = 90^\circ$ . Two circles with diameters  $AB$  and  $CD$  meet at points  $P$  and  $Q$ . Prove that  $2PQ < AD$ .

**Solution.** Let  $K, L$  be the projections of  $B, C$  to  $AD$ . Then the circles with diameters  $AB, CD$  pass through  $K$  and  $L$  respectively, and their common points  $P, Q$  lie on the arcs  $BK$  and  $CL$  (fig. 8.6). Since  $\angle BAK < 90^\circ$ , we

obtain that  $PQ \leq BK \leq AD/2$ , an equality is possible only when  $B$  and  $C$  coincide.

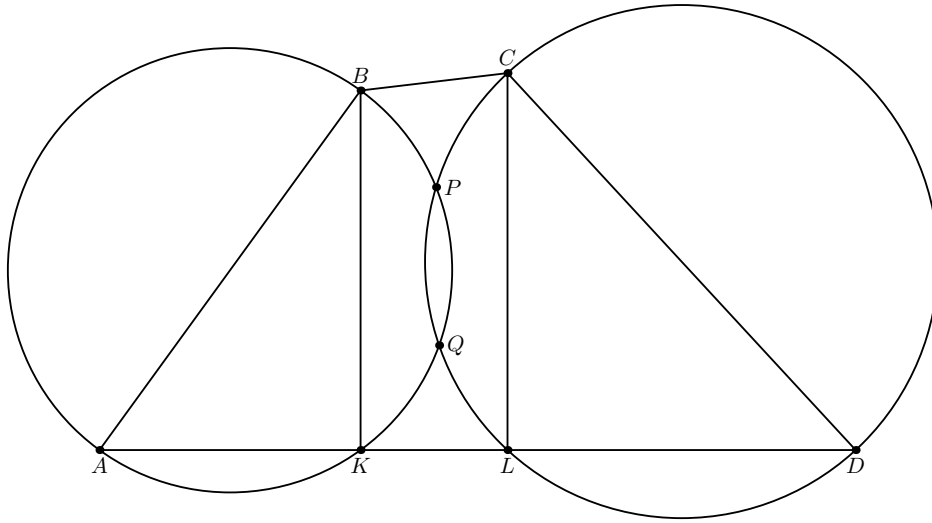


Fig. 8.6.

7. (K.Belsky) A regular triangle  $ABC$  is inscribed into a circle  $\Omega$ . Circles  $\Omega_A$ ,  $\Omega_B$ ,  $\Omega_C$  centered at  $A$ ,  $B$ ,  $C$  respectively pass through a point  $P$  lying on  $\Omega$  and have a common tangent. Prove that there exists a line touching two of these circles and passing through some vertex of  $ABC$ .

**First solution.** Let  $P$  lie on the arc  $AB$  of  $\Omega$ . Then by the Pompeiu theorem  $PC = PA + PB$ , hence the length of a common tangent to  $\Omega_A$  and  $\Omega_C$  equals  $\sqrt{AC^2 - (PC - PA)^2} = \sqrt{BC^2 - PB^2}$ , i.e. the length of a tangent from  $C$  to  $\Omega_B$ . Similarly the length of a common tangent to  $\Omega_B$  and  $\Omega_C$  equals to the length of a tangent from  $C$  to  $\Omega_A$ . Also, since  $\Omega_A$ ,  $\Omega_B$ ,  $\Omega_C$  have a common tangent, the length of a common tangent to one pair of these circles equals the sum of the lengths of common tangents to two remaining pairs. Thus we obtain the same equality for the lengths of tangents from  $C$  to  $\Omega_A$ ,  $\Omega_B$  and the length of a common tangent to these circles. Therefore one of common tangents to  $\Omega_A$ ,  $\Omega_B$  passes through  $C$ .

**Second solution.** Let  $\ell$  be the common tangent to  $\Omega_A$ ,  $\Omega_B$ , and  $\Omega_C$ . Supposing that  $P$  lies on the arc  $AB$  let us prove that the reflection of  $\ell$  about  $AB$  passes through  $C$ , i.e.  $C$  lies on the common tangent to  $\Omega_A$ ,  $\Omega_B$ . It is sufficient to prove that the reflection  $C'$  of  $C$  about the midpoint  $M$  of  $AB$  lies on  $\ell$ . Since  $PC = PA + PB$ , the distance from  $M$  to  $\ell$  equal to  $(PA + PB)/2$  is twice as little than the distance from  $C$  to  $\ell$ . Therefore reflecting  $C$  about  $M$  we obtain a point lying on  $\ell$  (fig. 8.7).

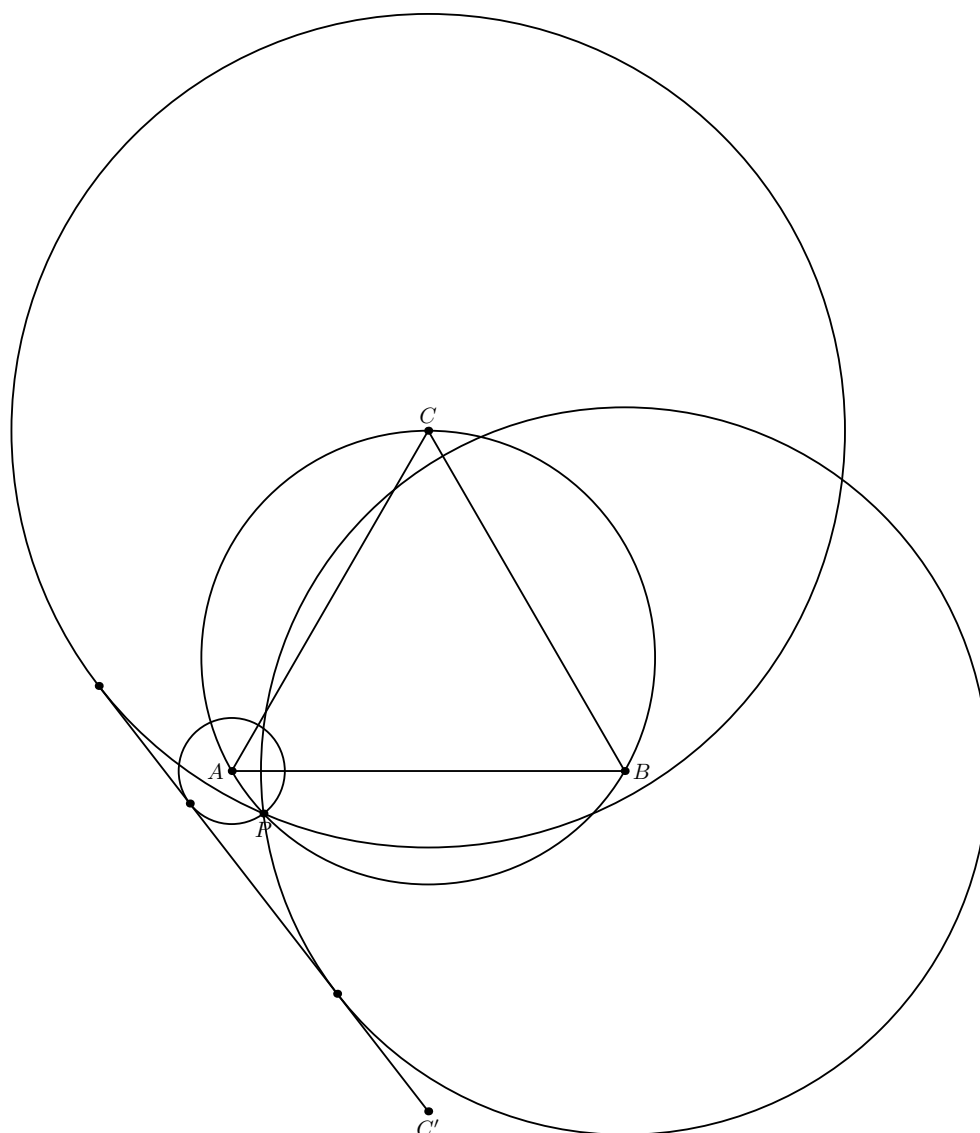


Fig. 8.7.

**Remark.** Any point  $P$  satisfying the assumption is a common point of the circumcircle and some excircle of the triangle. In this case three circles centered at  $A$ ,  $B$ ,  $C$ , and passing through  $P$  have a common tangent for an arbitrary triangle.

8. (A.Blinkov) Let  $ABCDE$  be a paper pentagon with  $AB = AE$ ,  $\angle A = \angle B = \angle E = 90^\circ$ ,  $BC = 3$ ,  $CD = 5$ ,  $DE = 2$ . Construct a perpendicular from  $A$  to  $CD$  using only a ruler and drawing not more than six lines. All lines have to be drawn inside the pentagon.

**Solution.** A circle centered at  $A$  with radius  $AB$  touches the lines  $BC$  and  $DE$ . Since  $CB + DE = CD$ , this circle touches also the line  $CD$ , i.e.  $CA$  and  $DA$  are the bisectors of angles  $C$  and  $D$  respectively, and  $\angle CAD = 45^\circ$ .

Let  $AK$  be the altitude of triangle  $ACD$ , and  $M, N$  be the common points of  $BE$  with  $AC$  and  $AD$  respectively. Then  $BC=CK$ ,  $DE = DK$ , and  $\angle CKM = \angle CBM = \angle DEN = \angle DKN = 45^\circ = \angle A$ . Therefore  $DM$  and  $CN$  are the remaining altitudes of triangle  $ACD$  (fig. 8.8). From this we obtain the following construction

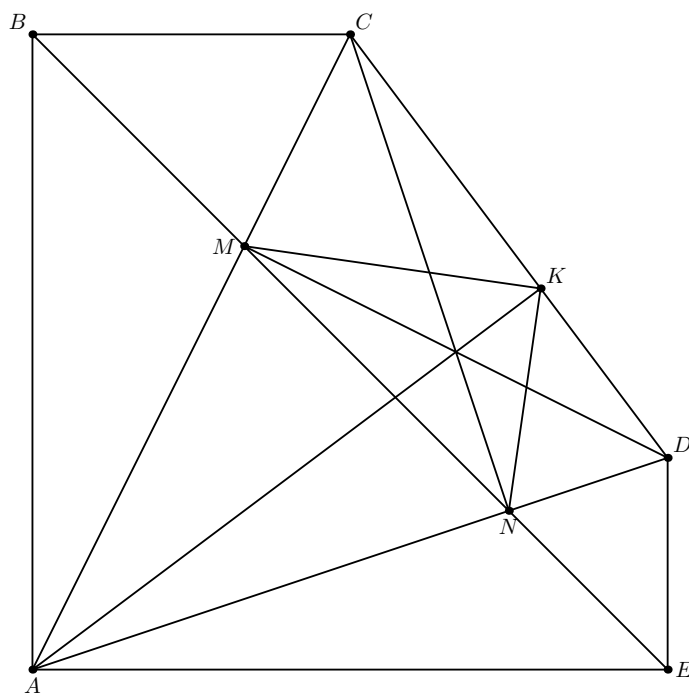


Fig. 8.8.

- 1–3. Draw the lines  $AC$ ,  $AD$ ,  $BC$  and mark the points  $M$ ,  $N$ .
- 4–5. Draw the lines  $DM$ ,  $CN$  and mark their common point  $H$ .
6. Draw the required line  $AH$ .

# XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

**Final round. First day. 9 grade.**

*July 31, 2025.*

1. (Ya.Scherbatov) Altitudes  $AA_1$ ,  $BB_1$  of a triangle  $ABC$  meet at point  $H$ . Let  $A'$ ,  $B'$  be the reflections of  $A$ ,  $B$  about  $BB_1$ ,  $AA_1$  respectively. Prove that the nine-points circles of triangles  $A'B'C$  and  $A'B'H$  are tangent.

**Solution.** Let  $K$ ,  $M_1$ ,  $N_1$ ,  $M_2$ ,  $N_2$  be the midpoints of  $A'B'$ ,  $CB'$ ,  $CA'$ ,  $HB'$ ,  $HA'$  respectively. Then we have to prove that  $\angle M_1KM_2 = \angle M_1N_1K + \angle M_2N_2K$ . Since  $\angle M_1KM_2 = \angle CA'H$ ,  $\angle M_1N_1K = \angle KB'C$ , and  $\angle KN_2M_2 = \angle HB'K$ , this is equivalent to the equality of angles  $CA'H$  and  $HB'C$ . But  $\angle HA'A = \angle A'AH = \angle HBB' = \angle BB'H = \pi/2 - \angle C$ , which yields the required equality (fig. 9.1).

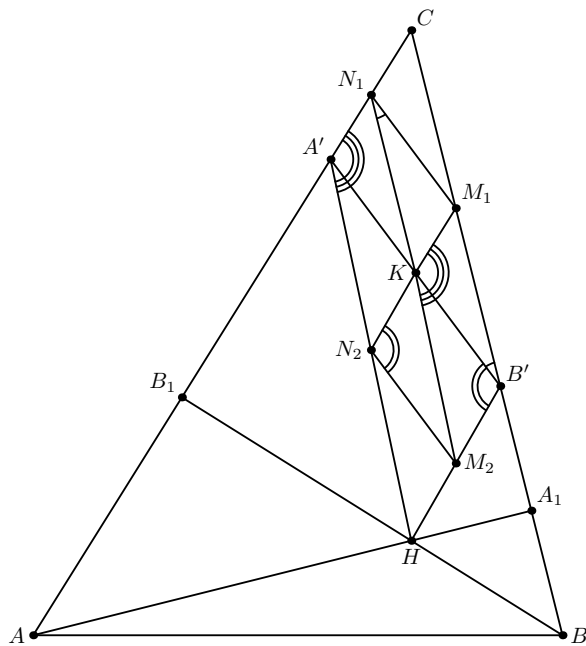


Fig. 9.1.

2. (F.Nilov) On the plane, several points are marked and colored into four colors so that any circle through three marked points of different colors contains exactly one marked point of the remaining color. Is it necessary that all marked points are concyclic?

**Answer.** No, it is not.

**Solution.** Consider the complete quadrilateral formed by lines  $A_1B_1$ ,  $A_1B_2$ ,  $A_2B_1$ ,  $A_2B_2$ . Let the lines  $A_1B_1$  and  $A_2B_2$  meet at point  $C_1$  and the lines



$A_1B_2$  and  $A_2B_1$  meet at point  $C_2$ . Then the circles  $A_1B_1C_2$ ,  $A_1B_2C_1$ ,  $A_2B_1C_1$ ,  $A_2B_2C_2$  meet at the Miquel point  $D_2$ . Make an inversion centered at an arbitrary point  $D_1$  not lying on constructed lines and circles, and color the maps of  $A_1$ ,  $A_2$  into the first color, the maps of  $B_1$ ,  $B_2$  into the second one, the maps of  $C_1$ ,  $C_2$  into the third color,  $D_1$  and the map of  $D_2$  into the fourth one. These eight points satisfy the assumption.

**Remark.** Another solution may be obtained from the addition of points on a circular cubic. Let  $A$ ,  $B$ ,  $C$ ,  $D$  be the common points of such cubic with an arbitrary circle, and  $K_1$ ,  $K_2$ ,  $K_3$  be three points of the cubic such that  $2K_i = 0$ . Then coloring  $A$ ,  $A + K_i$  into the first color,  $B$ ,  $B + K_i$  into the second one,  $C$ ,  $C + K_i$  into the third color, and  $D$ ,  $D + K_i$  into the fourth one we obtain 16 points satisfying the assumption.

3. (L.Emelyanov) A triangle  $ABC$  is given. A line  $m_1$  meets  $BC$ ,  $CA$ ,  $AB$  at points  $A_1$ ,  $B_1$ ,  $C_1$  respectively, and a line  $m_2$  meets  $BC$ ,  $CA$ ,  $AB$  at points  $A_2$ ,  $B_2$ ,  $C_2$ , so that  $A_1$  and  $A_2$  are symmetric about the midpoint of  $BC$ ,  $B_1$  and  $B_2$  are symmetric about the midpoint of  $CA$ ,  $C_1$  and  $C_2$  are symmetric about the midpoint of  $AB$ . Prove that  $m_1 \perp m_2$  if and only if  $m_1$  and  $m_2$  are two Simson lines of triangle  $ABC$  (for some two points of the circumcircle of  $ABC$ ).

**First solution.** The lines  $A_1B_1C_1$  and  $A_2B_2C_2$  generate a linear family of triangles  $A_tB_tC_t$ , where  $A_t$ ,  $B_t$ ,  $C_t$  divide the segments  $A_1A_2$ ,  $B_1B_2$ ,  $C_1C_2$  in the same ratio. This family contains the medial triangle  $A_0B_0C_0$  and two degenerate triangles  $A_1B_1C_1$  and  $A_2B_2C_2$ .

If two triangles of such family are orthologic, then two arbitrary triangles of the family are also orthologic. Hence  $m_1 \perp m_2$  yields that  $A_1B_1C_1$  and  $A_0B_0C_0$  are orthologic, i.e. the perpendiculars from  $A_1$ ,  $B_1$ ,  $C_1$  to the corresponding sidelines of  $ABC$  concur at some point  $P$ . Then  $P$  lies on the circumcircle of  $ABC$  and  $m_1$  is its Simson line. Similarly  $m_2$  is the Simson line of the opposite point on the circumcircle. Conversely, if  $m_1$ ,  $m_2$  are Simson lines, then  $A_tB_tC_t$  are pedal triangles of a linearly moving point  $P_t$ . Since this family contains the medial triangle, the corresponding line passes through the circumcenter  $O$ , thus  $m_1$  and  $m_2$  are perpendicular.

**Second solution.** Let the homothety centered at the centroid of  $ABC$  with coefficient  $-1/2$  map  $m_2$  to a line  $m'_2$ . Note that this homothety maps  $A_2$ ,  $B_2$ ,  $C_2$  to the midpoints of  $AA_1$ ,  $BB_1$ ,  $CC_1$ . Therefore  $m'_2$  is the Gauss line of the quadrilateral formed by the sidelines of  $ABC$  and  $m_1$ . It is known that

the Gauss line is perpendicular to the Simson line of the Miquel point, i.e. it is parallel to  $m_1$ . This is possible only if  $m_1$  coincides with the Simson line of the Miquel point. Since the Miquel point lies on the circumcircle of  $ABC$ , we obtain the required.

**Remark.** The Simson lines of any two opposite points satisfy the assumption (fig. 9.3).

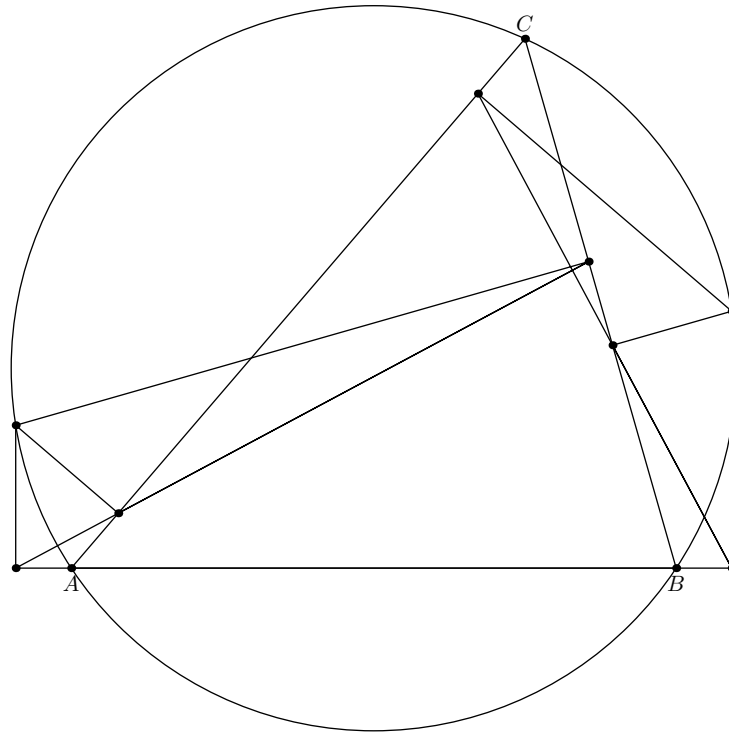


Fig. 9.3.

4. (E.Volokitin) Let  $ABCD$  be a cyclic quadrilateral. Two lines passing through the orthocenter  $H$  of the triangle  $ABC$  and parallel to  $BD$  and  $CD$  meet  $AC$  and  $AB$  respectively at points  $E$  and  $F$ . Prove that the line  $EF$  bisects the segment  $DH$ .

**First solution.** The midpoint of  $DH$  is the center of an equilateral hyperbola  $ABCDH$ . Let  $B', C'$  be the points of this hyperbola opposite to  $B, C$  respectively. Then  $HB' \parallel BD, HC' \parallel CD$ , and applying the Pascal theorem to the hexagon  $ABB'HC'C$  we obtain the required.

**Second solution.** Let  $O$  be the circumcenter of  $ABC$ . Note that  $\angle EHF = \angle A$ , hence there exists a point isogonally conjugated to  $H$  with respect to the quadrilateral  $BFEC$ , and this point coincides with  $O$ . Let  $X$  and  $C_1$  be projections of  $H$  to  $BF$  and  $EF$  respectively. Denote by  $M$  and  $C'$  the

projection of  $O$  to  $EF$  and the midpoint of  $CH$  respectively. Then  $M, C', X, C_1$  lie on the nine-points circle of triangle  $ABC$ , because the pedal circles of  $O$  and  $H$  with respect to  $BEFC$  coincide. From the Fuss lemma we obtain that  $FH \parallel C'M$ , thus the homothety centered at  $H$  with coefficient 2 maps  $M$  to  $D$ .

**Third solution.** Let  $H_b, H_c$  be the meeting points of altitudes of  $ABC$  with the circumcircle;  $Q_b, Q_c$  be the projections of  $D$  to  $AC, AB$  respectively;  $D_b, D_c$  be the meeting points of these perpendiculars with the circumcircle;  $P_b, P_c$  be their common points with the altitudes. Note that  $BDD_bH_b$  is an isosceles trapezoid, thus  $\angle DBH_b = \angle EH_bH = \angle EHH_b$ , i.e.  $HE \parallel BD$ , and  $E$  lies on  $H_bD_b$ . Similarly  $F$  lies on  $H_cD_c$ . Applying the Pascal theorem to the hexagon  $H_bD_bDACH_c$  we obtain that  $EP_b$  passes through the common point of  $AD$  and  $H_bH_c$  (fig. 9.4). Similarly  $P_cE$  passes through this point. Then by the Desargues theorem the triangles  $FP_cQ_c$  and  $EP_bQ_b$  are perspective, but  $P_bP_c$  bisects  $DH$  as a diagonal of parallelogram  $HP_bDP_c$ , and  $Q_bQ_c$  bisects it as the Simson line of  $D$ .

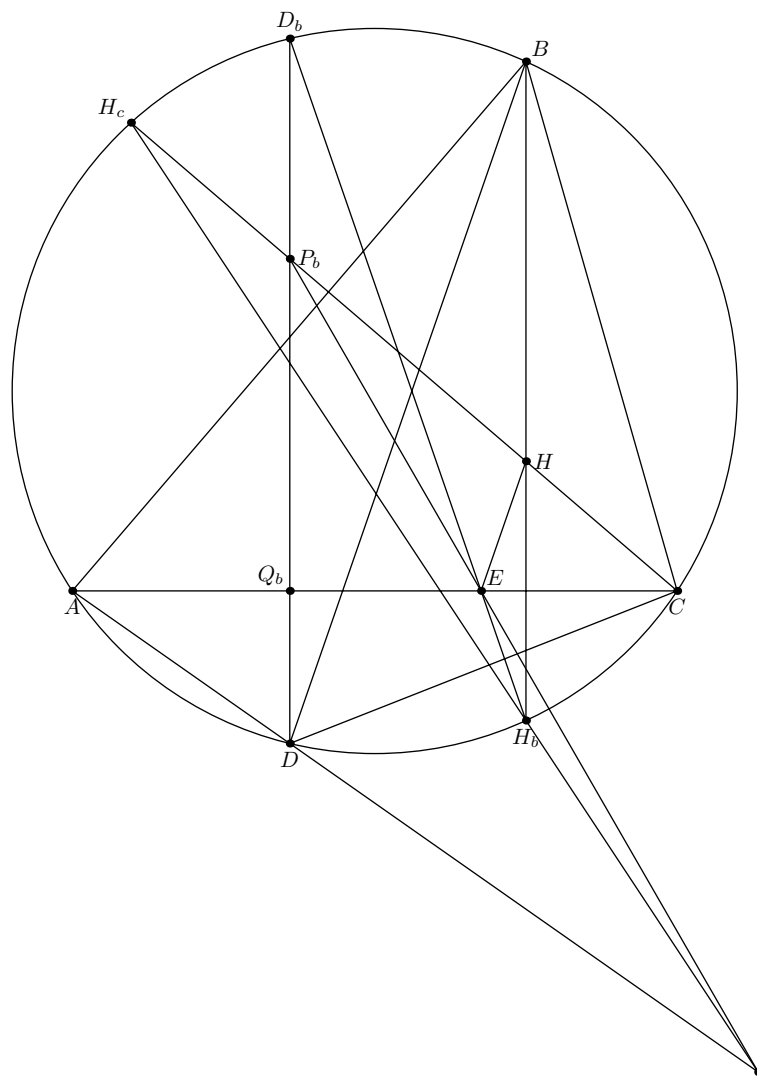


Fig. 9.4.

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## Final round. Second day. 9 grade.

August 1, 2025.

5. (I.Mikhailov) Let  $BE$  and  $CF$  be altitudes of a triangle  $ABC$ . The internal bisectors of angles  $B$  and  $C$  meet at point  $I$ , and the external ones meet at point  $J$ . Prove that  $IJ > EF$ .

**Solution.** By the trident theorem  $B$  and  $C$  lie on the circle with diameter  $IJ$ , therefore  $BC \leq IJ$ . On the other hand, from the similarity of triangles  $ABC$  and  $AEF$  we have  $EF = BC \cos \angle A < BC$ .

6. (ALEPH<sup>1</sup>/D.Brodsky) A triangle  $ABC$  is inscribed into a circle  $\omega$ . The tangents to  $\omega$  at points  $B$  and  $C$  meet at point  $S$ . The segments  $AS$  and  $BC$  meet at point  $P$ . The bisectors (the rays) of angles  $APC$  and  $SPC$  meet  $\omega$  at points  $X$  and  $Y$  respectively. Prove that  $X$ ,  $Y$ , and  $S$  are collinear.

**Solution.** Let us prove a general assertion: if a line passing through  $S$  meets  $\omega$  at points  $U$ ,  $V$ , then the quadruple of lines  $PA$ ,  $PB$ ,  $PU$ ,  $PV$  is harmonic.

Consider a projective transformation fixing  $\omega$  and mapping  $P$  to its center. It maps  $BC$  and  $AS$  to perpendicular diameters of the circle, and maps  $U$ ,  $V$  to points symmetric about  $BC$ . The required assertion is evident.

7. (ALEPH/N.Shteinberg) A triangle  $ABC$  is given. Let  $D$  be an arbitrary point on the perpendicular bisector to  $BC$ , lying outside the triangle. The lines  $BD$  and  $AC$  meet at point  $C'$ , and the lines  $CD$  and  $AB$  meet at point  $B'$ . A point  $M_a$  is the midpoint of  $BC$ , and  $M$  is the second common point of circles  $(BB'D)$  and  $(CC'D)$ . Prove that the circumcenter of  $DMM_a$  lies on a fixed line.

**Solution.** Let  $H_a$  be the common point of the  $A$ -Apollonius circle of  $ABC$  and the median  $AM_a$ . Let us prove that  $H_a$ ,  $D$ ,  $M_a$ ,  $M$  are concyclic, this clearly yields the required assertion.

Since  $M$  is the Miquel point of lines  $AB'$ ,  $CB'$ ,  $AC'$ ,  $BC'$ , we obtain that  $ABM \sim MDC$ , therefore  $\frac{MB}{AB} = \frac{MD}{DC} = \frac{MD}{DB} = \frac{MC}{AC}$ , i.e.  $M$  lies on the Apollonius circle.

Let  $A'$  be the reflection of  $A$  about  $BC$ ,  $A'$  also lies on the Apollonius circle.

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<sup>1</sup><https://aleph-problems.com/>

Let us prove that  $A'$ ,  $M$ ,  $D$  are collinear. When  $D$  moves along the perpendicular bisector to  $BC$ ,  $M$  projectively moves along the circle, thus it is sufficient to prove this for three positions of  $D$ .

If  $D$  is infinite, then  $M$  coincides with  $A$ , and the assertion is correct.

If  $D$  is the circumcenter of  $A'BC$ , then  $M$  coincides with  $A'$ , because the triangles  $A'DC$  and  $A'BA$  are similar. Also the tangent to the Apollonius circle at  $A'$  passes through  $D$ , because this circle is perpendicular to the circumcircle of  $A'BC$ .

If  $D$  lies on the circle  $A'DC$ , then  $A'D$  is the bisector of triangle  $BA'C$ , and  $M$  is the foot of this bisector because the triangles  $MDC$ ,  $MBA'$ , and  $MBA$  are similar.

Finally note that  $M_aD \parallel AA'$ , thus by the inverse Fuss lemma  $H_a$ ,  $A$ ,  $M$ ,  $D$  are concyclic (fig.9.7).



$$= \frac{\pi}{2} - \frac{\angle BAC - \angle CAD}{2} + \frac{\angle BCA - \angle ACD}{2} = \frac{\sphericalangle BC + \sphericalangle AD}{2} = \angle(BD, AC).$$

Similarly  $FI \parallel AC$ .

**Lemma 2.** Let  $AC$  and  $BD$  meet  $EF$  at points  $K, L$  respectively. Then  $\angle EIK = \angle FIL = \pi/2$ .

**Proof.** Let  $O$  be the circumcenter of the quadrilateral, and  $OI$  meet  $EF$  at point  $P$ . Since  $EF$  is the polar of the common point of diagonals, lying on  $OI$ , we have  $EF \perp OI$ , and  $P$  lies on the circle with diameter  $EI$ , touching the circle  $AIC$  by lemma 1. Also  $A, C, P$  lie on the circle with diameter  $OE$ . Thus  $K$  is the radical center of circles  $ACE, IPE, AIC$ , and  $\angle EIK = \pi/2$ . Similarly  $\angle FIL = \pi/2$ .

From these lemmas we obtain the following construction.

1. Draw the lines through  $I$  perpendicular to  $EI, IF$  and mark the points  $K, L$ .
2. Draw the lines  $k, \ell$  passing through  $K, L$  and parallel to  $IF, IE$  respectively.
3. Construct the orthocenter  $O$  of triangle  $IEF$  and the circle with diameter  $OE$ , it meets  $k$  at points  $A, C$ . Similarly construct the points  $B$  and  $D$  (fig. 9.8).

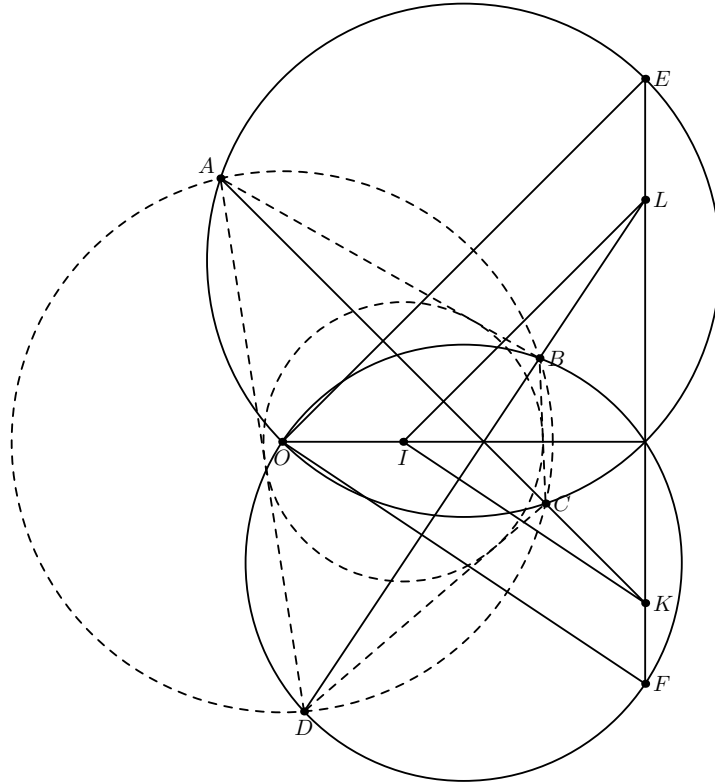


Fig. 9.8.



# XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

## Final round. First day. 10 grade.

July 31, 2025.

- (I.Bogdanov) Two parallelograms  $KL_1M_1N_1$  and  $KL_2M_2N_2$  are inscribed into a convex quadrilateral  $ABCD$  in such a way that  $K$  is the midpoint of  $AB$ ,  $L_1, M_1, N_1$  and  $L_2, M_2, N_2$  lie on the sides  $BC, CD, DA$  respectively. Can the area of one parallelogram be less than a half of the area of the quadrilateral, and the area of the second one be greater than the half of the area of the quadrilateral?

**Answer.** No, they cannot.

**Solution.** Suppose that  $\angle A + \angle B < \pi$ . Fix an arbitrary parallelogram  $KLMN$  such that  $N$  lies on the segment  $AP$ ,  $L$  lies on the segment  $BP$ , and  $M$  lies inside the triangle  $ABP$ , where  $P$  is the common point of the lines  $AD$  and  $BC$ . Let an arbitrary line passing through  $M$  meet the segments  $BP, AP$  at a points  $C', D'$  respectively (fig. 10.1). It is known that the area of triangle  $PC'D'$  is minimal, when  $M$  bisects  $C'D'$ . In this case  $KLMN$  is the Varignon parallelogram of the quadrilateral  $ABC'D'$  and its area equals a half of the area of this quadrilateral. Thus the areas of both parallelograms  $KL_1M_1N_1$  and  $KL_2M_2N_2$  are not less than the half of the area of  $ABCD$ .

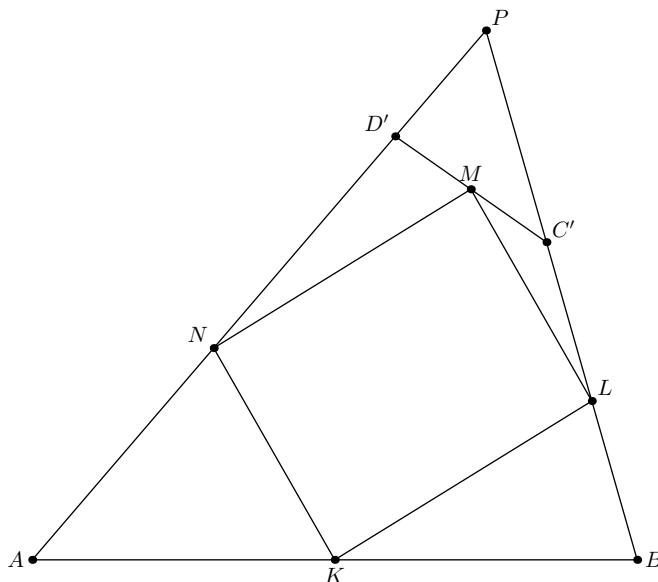


Fig. 10.1.

Similarly we obtain that if  $\angle A + \angle B > \pi$ , then the areas of both parallelograms are not greater than the half of the area of the quadrilateral. Finally, if

$AD \parallel BC$ , then the areas of both parallelograms are equal to the half of the area of the quadrilateral.

2. (E.Volokitin) Let  $I$  be the incenter of a scalene triangle  $ABC$ ; and  $P, Q$  be two isogonal points such that  $AP \parallel IQ \parallel BC$ . Prove that  $AP = |AB - AC|$ .

**Solution.** For any point  $P'$  such that  $AP' \parallel BC$  the sum of oriented areas of triangles  $P'AB$  and  $P'CA$  equals zero, and the ratio of these areas and the area of triangle  $P'BC$  equals  $\pm AP'/BC$ . Take a point  $P'$  such that the ratio of oriented areas is  $S_{P'BC} : S_{P'CA} : S_{P'AB} = a : (b - c) : (c - b)$ . Then for the isogonal point  $Q'$  we have  $S_{Q'BC} : S_{Q'CA} : S_{Q'AB} = a : b^2/(b - c) : c^2(c - b)$ , therefore  $S_{Q'BC} : S_{ABC} = a : (a + b + c)$ , i.e.  $Q'I \parallel BC$ . Hence  $P', Q'$  coincide with  $P, Q$ .

**Remark.** Here is an idea of another solution.

Let  $a, b, c, p$  be the sidelengths and the semiperimeter of  $ABC$ . Denote by  $X$  the second common point of  $IQ$  and the circumcircle of  $BIC$ . Then  $BIXC$  is an isosceles trapezoid. Therefore the projection of  $IX$  to  $BC$  equal  $|(p - b) - (p - c)| = |b - c|$ . Hence it is sufficient to prove that  $IX = AP$ . Let  $X'$  be isogonally conjugated to  $X$  with respect to  $ABC$ . Then  $X$  and  $X'$  are symmetric about  $AI$ , thus we have to prove that  $P$  and  $X'$  are symmetric about the perpendicular bisector to  $AI$ . Let  $\mathcal{C}$  be the isogonal map of  $IQ$  with respect to  $ABC$ . Then  $A, B, C, I, X', P$  lie on the hyperbola  $\mathcal{C}$ , and the lines  $AQ$  and  $IQ$  touch it. Since  $IQ = AQ$ , we obtain that the perpendicular bisector to  $AI$  is an axis of  $\mathcal{C}$ . Then the reflection about it maps  $P$  to  $X'$ .

3. (E.Alkin, A.Skopenkov) Do there exist six point  $A_1, \dots, A_6$  in general position in the space, such that the triangles  $A_1A_2A_3$  and  $A_4A_5A_6$  are linked, and two triangles corresponding to any other dividing of these points into two triplets are not linked. *Two triangles on the space are linked if the outline of one triangle meets the inside of the second one at a unique point.*

**Answer.** Yes, they do.

**Solution.** Take two congruent regular pyramids with common base  $A_1A_2A_3$  and vertices  $A_4, A_5$  symmetric about the base. Take point  $A_6$  on the plane passing through  $A_1A_4$  and the altitude of triangle  $A_5A_2A_3$ , inside the angle vertical to the angle containing  $A_1, A_4, A_5$  (fig. 10.3). Then all segments  $A_6A_i, i = 1, \dots, 5$ , lie outside the bipyramid  $A_1A_2A_3A_4A_5$ , and the points  $A_1, \dots, A_6$  satisfy the assumption.

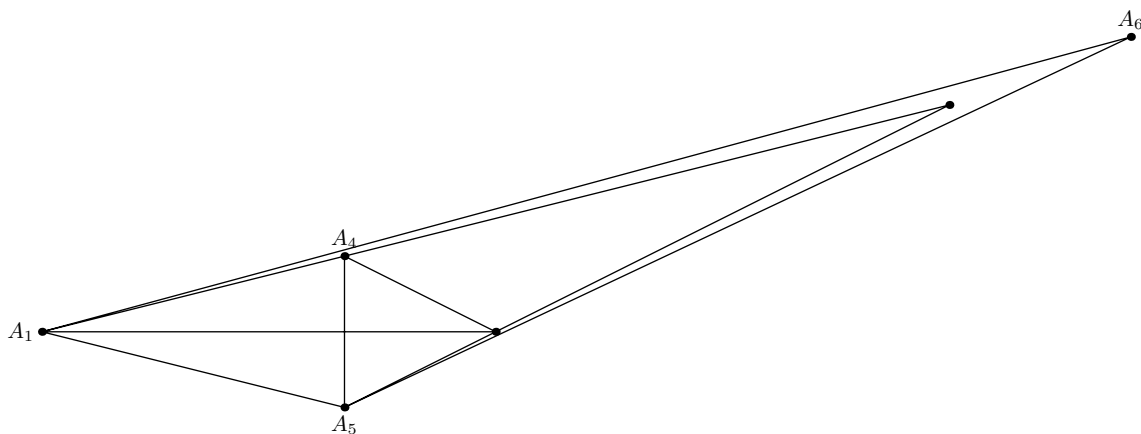


Fig. 10.3.

4. (S.Kuznetsov) Let  $M, H, L$  be the centroid, the orthocenter, and the Lemoine point respectively of a triangle  $ABC$ . A point  $S$  is such that the circles  $SLH, SML$  touch  $MH$ , and  $L'$  is the reflection of  $L$  about the circumcircle of the triangle. Prove that  $SL' \parallel MH$ .

**First solution.** Use the following property of an equilateral hyperbola.

Let points  $A, B$  lie on an equilateral hyperbola centered at  $O$ . The tangents to the hyperbola at  $A, B$  meet at point  $S$ . Then  $O$  is the Humpty point of triangle  $SAB$  corresponding to  $S$ .

In fact, construct a rectangle  $AUBV$  with the sides parallel to the asymptotes of the hyperbola. The points  $U, V, O, S$  are collinear and form a harmonic quadruple, therefore  $O$  and  $S$  are inverse about the circle with diameter  $AB$ .

Apply this property to the Kiepert hyperbola  $ABCMH$ . The tangents to it at  $M, H$  meet at  $L$ , therefore  $S$  is the center of the hyperbola, i.e. the midpoint of segment  $T_1T_2$  between two Torricelli points. On the other hand  $L'$  is the midpoint of segment  $T'_1T'_2$  between two Apollonius points. But  $T_1T'_1 \parallel T_2T'_2 \parallel MH$  (fig. 10.4).

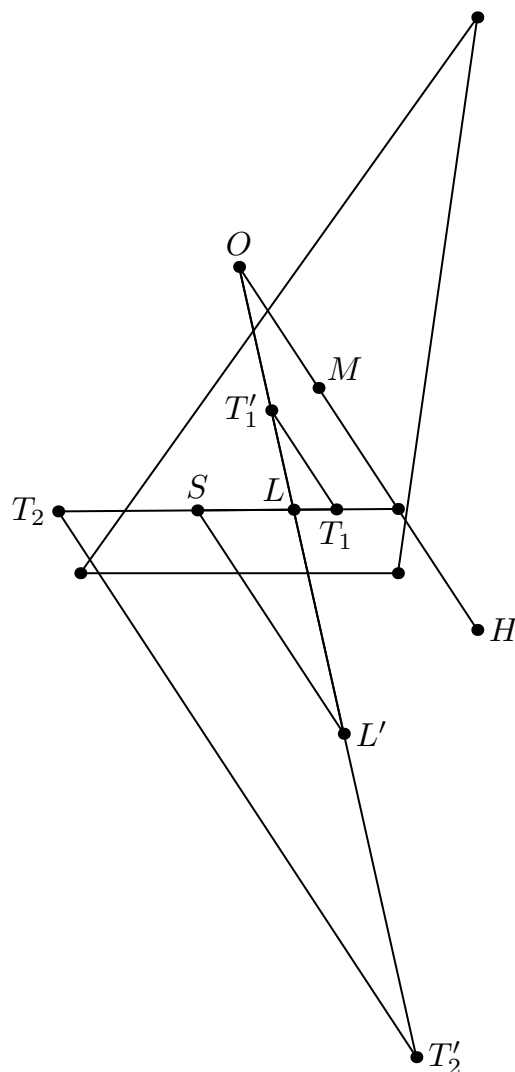


Fig. 10.4.

**Second solution.** Let  $\Gamma$  be an ellipse with foci  $O, H$  inscribed into  $ABC$ . Rotate  $ABC$  by Poncelet between its circumcircle and  $\Gamma$ . We have that  $H$  is the orthocenter of all triangles  $ABC$  as the isogonal image of the circumcenter  $O$ , hence the centroid  $M$  and the nine-points circle of  $ABC$  are also fixed. The point  $S$  lies on the nine-points circle, because it is the center of the Kiepert hyperbola, therefore its inversion image — the point  $L$  moves along a circle  $\Delta$  with a center lying on  $MH$ , and  $O$  is the reflection of the midpoint of  $MH$  about this circle. The point  $L'$  as the inversion image of  $L$  about the circumcircle of  $ABC$  also moves along some circle, and the angle velocities of  $L'$  and  $S$  are equal. It is easy to see that the radius of this circle equals the radius of the nine-points circle. Therefore  $SL' \parallel MH$ .

**Remark.** The points  $S$  and  $L'$  are symmetric about the circumcircle and the point  $M$ .

# XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

**Final round. Second day. 10 grade.**

*August 1, 2025.*

5. (K.Belsky) Let  $P$  be a random point inside a regular triangle  $ABC$ . Find the probability that there exists an acute-angled triangle with the sidelengths  $AP$ ,  $BP$ ,  $CP$ .

**Answer.**  $4 - \frac{2\pi}{\sqrt{3}}$ .

**Solution.** Consider a rotation by  $\pi/3$  about  $C$ , mapping  $A$  to  $B$ , and  $P$  to a point  $Q$ . We have  $BQ = AP$ ,  $PQ = CP$  (because  $CPQ$  is a regular triangle), therefore the sidelengths of triangle  $PBQ$  are equal to the lengths of segments  $AP$ ,  $BP$ ,  $CP$ . Also  $\angle PQB = \angle CQB - \pi/3 = \angle CPA - \pi/3$  (fig.10.5.1). Similarly two remaining angles equal  $\angle APB - \pi/3$  and  $\angle BPC - \pi/3$ . Thus this triangle is acute-angled if and only if all sides of  $ABC$  are seen from  $P$  at angles less than  $5\pi/6$ .

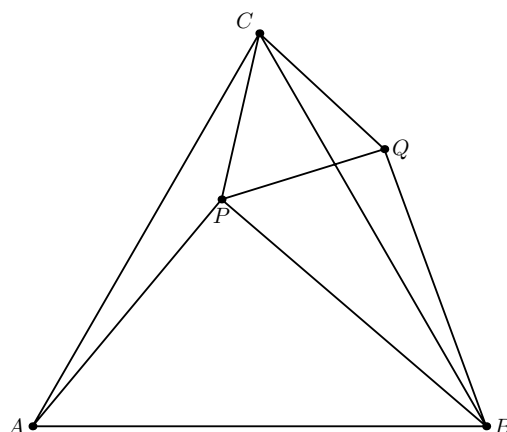


Fig. 10.5.1.

Let  $A'$ ,  $B'$ ,  $C'$  be the reflections of  $A$ ,  $B$ ,  $C$  about the opposite sidelines of  $ABC$ . Then  $AB$  is seen at the angle  $5\pi/6$  from the points of arc centered at  $C'$  with radius  $C'A = AB$ . This arc and two arcs constructed similarly for the remaining sides are pairwise tangent at the vertices of  $ABC$  (fig.10.5.2). Hence the area of a curvilinear triangle limited by these arcs equals the difference of the area of  $A'B'C'$  and the areas of three sectors with radius  $AB$  and angle  $\pi/3$ , i.e.  $\sqrt{3} - \pi/2$  (if  $AB = 1$ ). Dividing by the area of  $ABC$  we obtain the required probability.

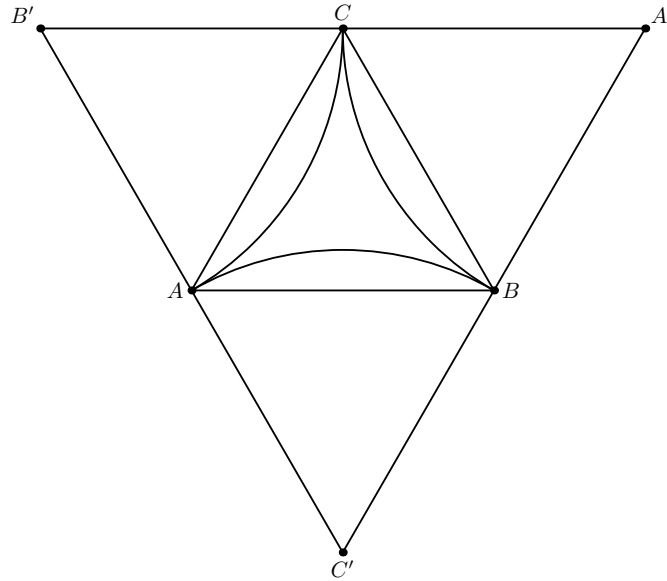


Fig. 10.5.2.

6. (Ya.Scherbatov) Circles  $\Omega$  and  $\omega_a$  are the circumcircle and the  $A$ -excircle of some triangle  $ABC$ . Let  $I_b, I_c$  be the centers of two remaining excircles, and  $A_b, A_c$  be the touching points of the extensions of  $AB, AC$  with  $\omega_a$ . Prove that the meeting point of the lines  $A_bI_b$  and  $A_cI_c$  does not depend on the triangle  $ABC$ .

**Solution.** Let  $\Omega$  and  $\omega_a$  meet at points  $P$  and  $Q$ , and the lines  $PQ$  and  $AB$  meet at point  $X$ . Since  $PQ$  is the radical axis of  $\Omega$  and  $\omega_a$ , and  $AB$  is the radical axis of  $\Omega$  and the circle  $I_bABI_a$ , we obtain that  $X$  lies on the radical axis of  $\omega_a$  and the circle with diameter  $I_bI_a$ , i.e. on the polar of  $I_b$  with respect to  $\omega_a$ . Thus the polars with respect to  $\omega_a$  of  $A_b$  and  $I_b$  meet on  $PQ$ , therefore  $I_bA_b$  passes through the common point  $Y$  of tangents to  $\omega_a$  at  $P, Q$ . Similarly  $A_cI_c$  passes through  $Y$ , and  $Y$  clearly does not depend on  $ABC$  (fig.10.6).

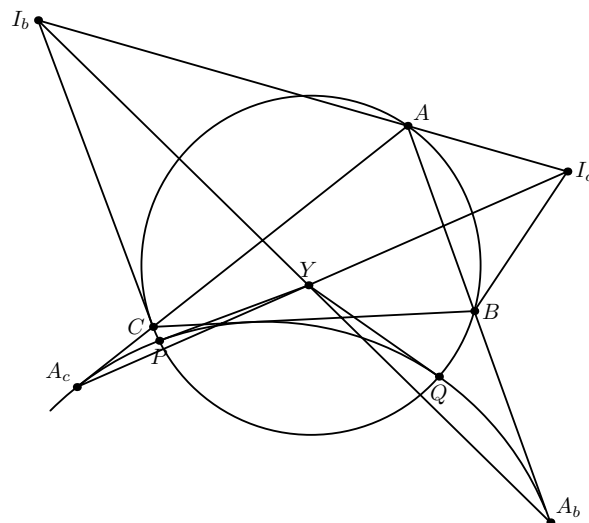


Fig. 10.6.

7. (M.Vekshin) Let  $ABCD$  be a cyclic quadrilateral. An arbitrary conic passes through  $A, B, C, D$ . Consider four lines which are the isogonal maps of this conic with respect to the triangles  $ABC, ABD, BCD, ACD$ . Prove that the quadrilateral formed by these lines is circumscribed.

**First solution.** An equation of an arbitrary conic passing through  $A, B, C, D$  is  $tF_1(x, y) + (1 - t)F_2(x, y) = 0$ , where  $F_1(x, y) = 0, F_2(x, y) = 0$  are the equations of two parabolas passing through  $A, B, C, D$ . The isogonal maps of all these conics with respect to  $ABC$  are parallel lines because the isogonal map of  $D$  is infinite. Clearly, when  $t$  changes uniformly, the corresponding line also moves uniformly. But the isogonal maps of circumparabolas touch the circumcircle of  $ABCD$ . Therefore the maps of any conic touch the same circle concentric with the circumcircle of  $ABCD$ .

**Second solution.** Denote by  $X$  and  $Y$  two infinite points of the given conic. Let  $X_a$  and  $Y_a$  be isogonally conjugated to  $X$  and  $Y$  with respect to the triangle  $BCD$ . Note that  $\widehat{X_a Y_a} = 2\angle(X, Y)$ , i.e. the length of this arc is the same for all four triangles. Since the isogonal maps of a conic cut off equal arcs, the distances from the circumcenter to these lines are equal.

**Remark.** If a conic is an ellipse, then  $X, Y, X_a, Y_a$  are imaginary points, but the equality  $\widehat{X_a Y_a} = 2\angle(X, Y)$  is correct.

**Third solution.** Prove that the isogonal images of any conic about the triangles  $ABC$  and  $ABD$  are symmetric about the perpendicular bisector to  $AB$ , this and similar assertions yield, that the distances from the circumcenter to all lines are equal. The isogonal images of the conics about each triangle

form a pencil of parallel lines. It is easy to see that the lines of both pencils form equal angles with  $AB$ , hence it is sufficient to prove that the corresponding lines meet at the perpendicular bisector. It is clear that the correspondence between the pencils is projective, and the infinite line corresponds to itself. By the Sollertinsky lemma it is sufficient to find two conics, such that their images meet on  $AB$ . These are two degenerated conics:  $AC \cup BD$  and  $AD \cup BC$ .

8. (K.Belsky) Let  $ABC$  be an acute-angled triangle. A line  $\ell$  meets the sides  $AB$ ,  $AC$  and the sideline  $BC$  at points  $C_1$ ,  $B_1$ ,  $A_1$  respectively. A circle  $\omega_a$  touches  $BC$  at  $A_1$  and touches the minor arc  $BC$  of the circumcircle of  $ABC$ . Circles  $\omega_b, \omega_c$  are defined similarly. Prove that these three circles have a common tangent.

**Solution.** Let  $A_2, B_2, C_2$  be the touching points of the circumcircle of  $ABC$  with  $\omega_a, \omega_b, \omega_c$  respectively. Then from the Archimedes lemma and the property of a bisector we obtain that  $\frac{BC_1}{AC_1} = \frac{BC_2}{AC_2}$ . By the Ceva theorem and the Menelaos theorem for  $\ell$  the lines  $AA_2, BB_2, CC_2$  concur. Take an inversion centered at  $A$ . We obtain the following problem.

The circle  $\omega_b$  touches the lines  $AC, BC$  at points  $B_1, B_2$ , and the circle  $\omega_c$  touches  $AB, BC$  at points  $C_1, C_2$  respectively. Let  $A_2$  be a point on  $BC$  such that  $A_2C \cdot A_2C_2 = A_2B \cdot A_2B_2$ . Let us prove that there exists a circle touching  $\omega_a, \omega_b, \omega_c$  and passing through  $A$ .

Construct a circle  $\omega$  passing through  $A$  and touching internally  $\omega_b$  and  $\omega_c$ . Let us prove that  $\omega$  and  $\omega_a$  are tangent. Let  $\omega$  meet  $BC$  at points  $X$  and  $Y$ . Let  $I_1$  and  $I_2$  be the incenters of triangles  $AXY$  and  $ABC$ . By the Sawayama lemma  $B_1B_2$  and  $C_1C_2$  meet at point  $I_1$ . Then the triangles  $BI_2C$  and  $B_2I_1C_2$  are homothetic. This yields that  $A_2$  lies on  $I_1I_2$ . Applying the inverse Sawayama lemma to the tangent from  $A$  to  $\omega_a$  and the line  $BC$  we obtain the required (fig. 10.8).



Fig. 10.8.