The Structure of Operator Algebras

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Course Description

This intensive 8-week course provides a rigorous introduction to the foundational theory of C*-algebras, which form a bridge between analysis and algebra. Starting from the concrete geometry of Hilbert spaces, we will study bounded linear operators and then abstract their essential properties to define C*-algebras. The central goal of the course is to understand the two monumental Gelfand-Naimark theorems, which show that every C*-algebra can be realized concretely, either as an algebra of continuous functions on a topological space or as an algebra of operators on a Hilbert space.

Prerequisites

A strong foundation in Linear Algebra and an introductory course in Real Analysis (covering metric spaces, completeness, and continuity).

Primary Text

G. J. Murphy, *C*-Algebras and Operator Theory*. We will refer to this text as **[M]**.

Weekly Course Schedule

The course is structured in two parts, closely following the first three chapters of **[M]**.

Part I: The Concrete World of Operators (Weeks 1-3)

Week 1: The Geometric Setting: Hilbert Spaces

- Inner products, completeness, and the definition of a Hilbert space.
- Orthogonality, orthonormal bases, and the Riesz Representation Theorem.
- The isomorphism of any separable Hilbert space with the sequence space ℓ^2.
- **Reference:** [M] Chapter 1.

Week 2: The Algebra of Bounded Operators

- Bounded linear operators on a Hilbert space, the operator norm.
- The algebra $B(H)$, the adjoint of an operator.
- The operator “zoo”: self-adjoint, normal, unitary, and projection operators.
- **Reference:** [M] Chapter 2 (Sections 2.1-2.2).

Week 3: The Spectrum

- The resolvent set and the **spectrum** of an operator, $\sigma(T)$.
- Key properties: the spectrum is non-empty and compact.
- Calculation of spectra for key examples (diagonal operators, the shift operator).
- **Reference:** [M] Chapter 2 (Section 2.2).

Part II: The Abstract and its Powerful Structure (Weeks 4-8)

Week 4: The Leap to Abstraction: C*-Algebras

- Banach algebras and the definition of a **C*-algebra**.
- The C*-identity: $\|x^*x\| = \|x\|^2$.
- The two guiding examples: the non-commutative algebra $B(H)$ and the commutative algebra $C(X)$.
- **Reference:** [M] Chapter 2 (Sections 2.1, 2.3).

Week 5: The Commutative World: The Gelfand Transform

- Characters (multiplicative linear functionals) and the character space $\Omega(A)$.
- The Gelfand Transform: $\Gamma : A \to C(\Omega(A))$.
- **Reference:** [M] Chapter 2 (Section 2.4).

Week 6: The Commutative Gelfand-Naimark Theorem

- Main Result: Every commutative C*-algebra is isometrically *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.
- The profound ”algebra-is-topology” dictionary.
- **Reference:** [M] Chapter 2 (Section 2.4).

Week 7: States and Representations

- Positive elements, positive linear functionals, and states.
- The concept of a *-representation: $\pi : A \to B(H)$.
- The Gelfand-Naimark-Segal (GNS) Construction: turning a state into a representation.
- **Reference:** [M] Chapter 3 (Sections 3.1-3.4).

Week 8: The General Gelfand-Naimark Theorem & Synthesis

- Main Result: Every C*-algebra is isometrically *-isomorphic to a C*-subalgebra of $B(H)$ for some Hilbert space H.
- Course review: The full circle from concrete operators to abstract algebras and back.
- **Reference:** [M] Chapter 3 (Section 3.4).

Optional Final Projects

Students interested in exploring topics beyond the core syllabus may undertake an optional final project. The goal is to read, synthesize, and present the main ideas of a more advanced chapter from the textbook. The following three projects are independent of one another.

Project 1: Von Neumann Algebras and the Bicommutant Theorem

- **Description:** This project introduces von Neumann algebras, which are C*-algebras with a richer structure tied to measure theory. The central goal is to understand the different topologies on $B(H)$ (weak and strong) and to state and explain the significance of von Neumann's spectacular Double Commutant Theorem, which gives a purely algebraic characterization of these algebras.
- **Source:** [M] Chapter 4.

Project 2: Tensor Products of C*-Algebras and Nuclearity

- **Description:** This project tackles the question of how to combine two systems described by C*-algebras. This requires defining a C*-norm on the tensor product $A \otimes B$. You will explore the fact that there is no unique way to do this, leading to the minimal and maximal tensor products, and the crucial modern concept of *nuclearity*—a "niceness" property for a C*-algebra.
- **Source:** [M] Chapter 5.

Project 3: An Introduction to K-Theory for C*-Algebras

- **Description:** This project introduces a powerful tool from algebraic topology used to classify C*-algebras. The idea is to associate an algebraic invariant (an abelian group, $K_0(A)$) to a C*-algebra A. You will learn how this group is constructed from projections and how it serves as a "fingerprint" to distinguish non-isomorphic algebras, forming the foundation of the modern classification program.
- **Source:** [M] Chapter 6.

Project Deliverables

For any chosen project, the deliverables will be:

1. A written report (5–8 pages) summarizing the key definitions, theorems, and examples.
2. A brief in-class presentation (10–15 minutes) during the final week of the course.