AN INTRODUCTION OF A MATHEMATICAL DIARY

Souradip Das

February 1, 2022

Session 1

About Myself: Hello! I am Souradip Das, a student of class 9. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in that particular session.

1 Introduction

We had 6 students in the session. They all were 4th graders and classmates from the same rural primary school. Their School Teacher also took a class with us, since they had to take some time learning the new facilities in the online platform, which were new to them.

We had discussed 3 problems, 2 of them were puzzles, and 1 of them required some thinking. Note that some Exercises will also be given to the reader, if they want to try something from the diaries!

We started with our very known tic tac toe game.

2 The Tic Tac Toe

Tic Tac Toe is a very famous board game, and everyone has played it at least once in their childhood with their friends. As the rules of tic tac toe, 2 persons alternatively put crosses and circles on a 3x3 board, and whoever gets a 3 in a row, first, wins the game. Everyone had seen this game before, so it was not difficult for me to explain the game.

I started by posing a question, "Is there a way to always win this game? If that is not possible, then is there a way to not lose this game?". I wanted to ask if there was a good "strategy" to play this game, but I posed the question a bit differently.

Nobody answered anything, but that had made them think for some time. To make them think, I had decided to play 3 original tic tac toe games. After we opened our Jam Board (due to technical issues the students couldn't join the Jam Board for some time), I played 2 games with their school teacher. Both the games were a draw.

(The students were from rural regions, and so they were a bit new to the idea of the Jam Board. They were a bit excited about the board and started drawing figures and shapes, which took some more amount of time, but it was for fun :) !!)

After that, I played with a student named Ayesha. She started first, and that game also ended in a draw.

Then I started with a question, "Suppose 2 people play a game of tic tac toe, where I start the game. Who has a greater chance of winning? Me or My Friend?".

They were quiet for sometime, till their school teacher insisted them to communicate with me.

Then a lot of them answered (at once), that I would have more chances of winning. The answer was correct, but then I asked, "Why is that so? Is there any Specific Reason for which I have more chances of winning?"

One of them volunteered, and said, "First, you shall start the game, and then that becomes your first turn. Then suppose I play with you, and I give my next turn, and so on. If we go like this, then it may happen sometimes, that you would make 3 in a row, and if you would do that, it can also happen that my last turn won't occur, so in that way, you seem to have some advantage."

This was a different way of claiming this answer, and it's not wrong at all. The student thought about it in some different way, and then came up with it. Clearly she is right in saying that the player who starts first may make a 3 in a row at some point of time, and so it's possible that the 2nd player may not get a chance. She couldn't explain it properly, but she had the idea. In other words, Player 2 seems to have less chances to play in the board, than Player 1, and that is what gives Player 1 some advantage.

Then I cleared up the idea. I said, "Everyone, notice that in all the 3 games, which were all draws, the person who starts first (the one who gives the crosses), seems to be giving 5 of them, but the person starting second, seems to be giving 4 of them"

"Why is this so? This is because there are 9 places, and you cannot divide 9 equally into two equal parts, and the first player, starting first, gets an extra place. This clearly is an advantage for the first player. He gets more chances to control the board."

So, was there any way to figure out a strategy? One student, Arqam, suggested to play another game. I agreed upon it, but this time I said I would play with myself, to get an idea of all the possible positions and how to play wisely in each of them.

In Mathematics, this is also known as Casework. There are lots of problems where finding the answer using formulas may become hard, but using some Clever Casework, and Trial and Error, can overcome a lot of these problems.

So I started. "Assume my Friend and I are playing a game, now we both can start very differently, and we are going to analyze every of these games." "For now, assume that I start the game, and my friend starts next. Both of us will try our best to win the game, i.e. , no one will give any wrong moves."

O1		
X1		

(I am representing then as $X1$ and $O1$, to show the order in which the X's and O's are played.) "Suppose we start with this configuration, and now it's my turn. Where should I place my next X?"

		X2
O1		
X1		

One student suggested a move, to place it in the top right corner, but the game quickly turned to a draw. (**Exercise** Left to the Reader, find out why the game will be a draw.)

A big technique to win tic tac toe games is to make 2 consecutive 2 in a line, at once. Since the opponent cannot block both those lines in 1 move, a 3 in a row is for sure going to come, and then we win. Another technique is to always make forcing moves to the opponent, so that the opponent gets even less flexibility in playing on the board, and in some cases, this may even force a win!

Since the students (and almost all people) play tic tac toe, they are aware of these techniques and so it did not take much time for me to explain these to them.

Then, I claimed that there was a way to make my friend lose this game. In fact, there were exactly 3 different ways to force out a win from this position.

It did not take time for them to find them out. And it was pretty fun. (As an **Exercise**, find all the possible positions of the X_2, which forces out a win for Player 1.)

So, one case was completed. We analysed one case and saw that there were 3 forcing winning positions for me (the first player). Now we started analysing all the other positions.

I had also tried to show them the idea of symmetry by asking them, "If we had started with this :-

X1	O1	

"Then, would it make any difference?". It did not take time for them to realise that in this case the whole configuration just gets flipped/rotated in some direction, but it happens due to symmetry.

Onto the next case, what if we started with this configuration?

		O1
X1		

They thought for a bit, then Arqam said, "Put the X_2 in the top left corner. This also fixes up the O_2". So I do that :-

X2		O1
O2		
X1		

I claimed, "In this position, there is exactly 1 move (for X_3) which forces out a win for myself." (**Exercise**:- Find that move.)

They had found it. And since we had found 1 solution, where my friend (Player 2) had to play all forced moves, it seems that we can always win in this configuration.

(**Note** :- Since this board is symmetric, there will be 2 solutions in total, but each of them are symmetric, so there is just 1 unique way to force out a win from this configuration.)

Into the next Configuration. We can start like this :-

	O1	
X1		

There are exactly 3 unique ways to force out a win from here, and the students had found it after checking some moves and playing them. (**Exercise**:- Find all the ways to force out a win from the opponent.)

Once again I pull up the idea of symmetry. I said, "Notice, if we had taken this configuration" :-

		O1
X1		

"Then also, it seems that the whole game just rotates itself in some position, this happens due to the reflecting position of O_1, almost similar to one of those earlier reflecting cases." They agreed with it.

(For now, they had somewhat got the idea of how symmetry works, like a pattern. It was also somehow obvious to me, that we could have taken any corner for the X_1, due to symmetry, it wouldn't change the cases, but I forgot to mention it, and I should have done that.)

Next with almost all the cases checked with a cornered X_1, I posed the Question :- "We saw a lot of cases, and in all of those cases, we found a way to let my opponent make forcing moves, and managed to find out a win. So, shouldn't it be the case that in all tic tac toe games, the first player should always win? (Only if he plays wisely), and the second player should always lose? If that was the case, why were the first 3 games (those first 2 games I played with the school teacher), were draws? Is it because I was not playing wisely? Or because there is some reason to it?."

I was expecting two different kind of answers. One was that I have not yet checked the case when O_1 is at the center, and the other one was that we still have not checked those cases when X_1 is at a different position, especially not at the corners. (It can be at an edge, or at the center as well.) But no one responded to my question, so I moved on :-

"We still have not checked the case when O_1 is at the center, so let's check that first :-

	O1	
X1		

Now this case was tricky. Everyone tried for some time to find a win, but eventually all the games turned to a draw. We had a lot of discussions, but at the end, it seemed like all the games turned to a draw. One of the students claimed, "I think there is no way to win this game." Indeed, there is no win to this game. But I did not want to give any proof/intuition to why this would happen, so that they can think about it more, why there is no such win in this game. (As an **Exercise**, you can also try proving it!)

So, we arrived at a break. We saw that we start our X_1 from the corner, all was well (we were able to win almost all the games), till the case when the opponent makes his move in the center, which can at most give a draw to us.

But we seem to have some more cases in our bag. What if we started X_1 from the center?

Now, we seem to have some more different types of cases popping up. We will analyze each of them at once. (Note that by symmetry, we would have only 2 cases, and I hopefully managed to convince them about it.)

First, we can have O_1 at an edge.

O1	X1	

There are exactly 4 different ways to win this game, and 2 unique ways to win this game. The students had found 1 of them, and another one was obvious from the symmetry. (**Exercise**:- Find all the 4 winning moves for X_2)

Now, we can have O_1 at a corner.

O1		
	X1	

Now, we again face the same problem. We played a lot of moves, replayed all the moves, but it again seems that all the games turned out to be a draw. This was an observation for them, and I once again provide no proof/intuition for it. (**Exercise**: Prove it!)

That was all, that we had discussed in the Session about the first game. It was really fun, the students were interacting a lot when we were playing the games, when we were checking all the various possible cases, and so on. Next we will talk about the second game, in our next document.

Thank You for Reading!

(Note: I couldn't discuss one part of this, in the class, it is when X_1 started on an edge. I did not have the time to do it, and I also forgot about it. However it was good that I did not discuss it, since in most of those cases, the games will turn out to be draws. We can let them think about them why it is happening so.)

(**Exercise**: Analyze the case when X_1 is at an edge. What do you observe? Can you prove it?)

3 A Basket of Mangoes

We now started playing a new and a different type of game.

"There are 23 mangoes in a basket. 2 players play a game with the mangoes, alternating their turns. In each turn a player can either remove 1 or 2 mangoes from the basket. The person to remove the last mango, loses. Is there a way to win this game?"

To let them understand the game, I took an example. "Suppose my Friend and I are playing this game. I started removing the mangoes. Suppose I removed 1 mango, then there are 22 mangoes left. Then My Friend removed 2 mangoes, so then we would have 20 mangoes left, and so on. We can form a sequence like

$$23 \to 22 \to 20 \to 18 \to ...$$

You all can see that the amount of mangoes are slowly decreasing in the basket." They had understood upto this.

Then I posed the Question, "Is it possible to somehow, play this game wisely and eventually win it? Is it impossible? Or is it possible only in a certain cases?"

A lot of people shouted (and answered) at once, that it is always possible to win this.

The students wanted to play one or two games like that. So the first game was between Arqam and Sumana. Arqam started the game. I will denote A for Arqam's move at that turn, and S for Sumana's move at that turn.. This was the result :-

$$23 \to 21_A \to 19_S \to 18_A \to 17_S \to 15_A \to 14_S \to 12_A \to ....$$

Meanwhile, in between the game, I asked, "Are you all able to guess who is going to lose this game?" One student replied, "Sumana is going to lose!", then she thought for some time, and asked about the person who started the game. I reminded that Arqam had started the game by picking up 2 mangoes.

Then the student said, ”It depends on the game. Let’s see who wins and who loses.

Okay, continuing with the game, this was the result :-

$$12 \to 11_S \to 10_A \to 9_S \to 7_A \to 6_S \to 4_A \to 2_S \to 1_A$$

So, Arqam won the game.

Then I suggested for another game with me. A lot of people volunteered to play, so Jesmine played a game with me this time. She started the game. I will denote S for my move at that turn, and J for Jesmine’s move at that turn. This was the result :-

$$23 \to 21_J \to 19_S \to 17_J \to 16_S \to 15_J \to 13_S \to 12_J \to$$

$$\to 10_S \to 8_J \to 7_S \to 6_J \to 4_S \to 3_J \to 1_S$$

So I won the game. Now I asked them, ”Is there any reason of why I won this? Or was this just luck? Do you see any idea of how can this be played?

Ayesha claimed, ”When there were 4 mangoes left, if Jesmine had removed 2, then...” Then I said that I would have removed 1, so I would have still won. Everyone agreed.

Jesmine then tried by saying, ”I think in the 4th step, if I had removed only 2 mangoes, then there would be 5 left, then I would have won.” then I reminded, ”Then I could have removed 1 mango in my step, which would bring down to 4 mangoes, and we already saw that if there were 4 mangoes, it was a lose for you. So you would have still lost.” She agreed.

So I went to a new board, and said :- ”Imagine we just started with 1 mango, and start from backwards. We must have arrived to 2 or 3 mangoes to reach 1 mango, now to reach 2 mango, we must have arrived to 3 or 4 mangoes, and to reach 3 mango, we must have arrived to 4 or 5 mangoes, and so on ...

But, this does not seem like a nice way to do this. We are getting too many ways to reach 1 mango, so that it is impossible to do all these by hand. So we try looking for a simpler way.”

”In the previous game (between me and Jesmine), when Jesmine had 4 mangoes in the end, she lost, however she played. We all saw that.”

Jesmine said, ”So whoever who started first, with 4 mangoes, loses.” I asked her why.

She thought for a moment, and then replied, ”I think we can check each of the possible games from 4 and then conclude from there.” And she was right.

I said, ”Suppose Ayesha and I are playing the same game with 4 mangoes, and I start the game. These are the two possible games we can have with 4 mangoes. Either I remove 1 mango from there, or I remove 2 mangoes from there.

These are 2 possible games we could have :-

$$4 \to 3_S \to 1_A$$

$$4 \to 2_S \to 1_A$$

So, in both the games, I seem to lose. So, if a person started with 4 mangoes, then he is for sure losing the game." Till here, everyone accepted the claim.

Now, I said, "Do you all notice anything which is happening with 4 mangoes? If I remove 1 mango, then Ayesha removes 2 mangoes, but if I do the opposite, i.e. , if I remove 2 mangoes, then Ayesha removes 1 mango. And I am losing both of these games. So, if you imagine playing 2 turns at once, like suppose in one turn Ayesha and I are taking out mangoes, then notice that in each such turn, we are removing 3 mangoes."

They were taking some time to understand this, so I said, "The order in which we are removing mangoes, from 4 mangoes, is either $(1, 2)$ or $(2, 1)$, that is, either way, we are in total, removing exactly 3 mangoes, because $(1 + 2) = (2 + 1) = 3$, and that is just following some random order." Hopefully I felt that they have understood till here.

"Now, we are getting an idea here. From 4 we are going to 1 (the losing point), by removing 3 mangoes, so what if I had started with 7 mangoes?"

"Again, I would have 2 different ways to start it. But, now we all know (and we can assume Ayesha knows this too), that if SHE can make ME reach 4 mangoes, then I am definitely going to lose."

"But Hey! It seems that starting from 7 mangoes, Ayesha has a way to make me go to 4 mangoes, again by "removing the 3" technique."

$$7 \to 6_S \to 4_A$$

$$7 \to 5_S \to 4_A$$

"So, if I had started with 7 mangoes, Ayesha has a way to make me go to 4 mangoes, but we already know that the person starting with 4 mangoes, loses! Which means, we found a way to figure out, that the person, starting with 7 mangoes, also loses!!"

The students were taking some time in fully understanding this, so they asked me to repeat it. I said "Ok, suppose for now, assume the game started with 7 mangoes, then we can see from there, that Ayesha has a way to make me go to 4 mangoes, but why 4? Why not 5 or 6? 4 is important because that is also a number where we figured out, that whoever starts with 4 mangoes, also loses. And Ayesha has a way to make her turn such that she can FORCE me to reach 4 mangoes, but that's over! That surely means I will lose. So, I will have to lose with 7 mangoes as well!!"

Then I said, "Now, do you notice any pattern with these numbers, $1, 4, 7, ...$? These are the numbers we know, that whoever starts with these amount of mangoes, is going to lose this game."

Arqam then said, "The next number will be 10."

I said, "Exactly. then comes $13, 16, 19$ and so on. These are the numbers, which we can call as the "Losing Numbers" for the first player. Why is it so? Because that's exactly how the pattern seems to move. We all saw how 7 became a losing number, since it reaches to 4, which again reaches to 1.

Similarly, continuing the same pattern, we will get that $10 \to 7 \to 4 \to 1$. This goes on with $13, 16, 19$ and so on. Also notice how this numbers are getting added with 3."

"So, whoever starts with a number of mangoes, and this number is in that list, we can guarantee that, that person is going to lose. (Provided the opponent plays wisely, of course.)"

"Notice, $1, 4, 7, 10, 13, 16, 19, 22, 25, ...$, we have also got the number 22 in our list. This means that whoever starts with 22 mangoes, is surely going to lose this game. And we have 23 mangoes at the start. To see how this works, let's play a final game."

This time Ayesha volunteered to play the game. And this was the game we played :-

$$23 \to 22_A \to 20_S \to 19_A \to 17_S \to 16_A \to 15_S \to 13_A \to 12_S \to$$

$$\to 10_A \to 8_S \to 7_A \to 5_S \to 4_A \to 2_S \to 1_A$$

So, Ayesha won the game!

There were some wrong moves (made by Ayesha), which could have made me win, but I cleared the doubts, by saying, "We just remember this numbers, "{1,4,7,10,13,16,19,22,25,...}". Whoever starts with any one of these no. of mangoes, loses. Like in the move when we had 15 mangoes, if you had removed 1 mango, there would be 14 mangoes in my turn, but then I can remove 1 mango from there and then there will be 13 mangoes in Ayesha's turn, but 13 now becomes a losing number for Ayesha (which was for me before)."

(**Note**:- In Game Theory, this is called stealing the strategy, which is also an important topic in many such other games.)

(**Exercise**: Suppose my Friend and I are playing this game, but this time, my friend starts first. Do I still have a way to win this game? If yes, find out the largest number of mangoes < 100 that I can possibly start with, such that I win this game, no matter how my friend plays.)

That was all, that we had discussed in our second game we played. The students, as usual, were very enthusiastic and were more interested in playing the games with others (for fun), but they still did listen and paid attention to why this game works, how there seems to be a way to beat this game. In other words, there was a strategy.

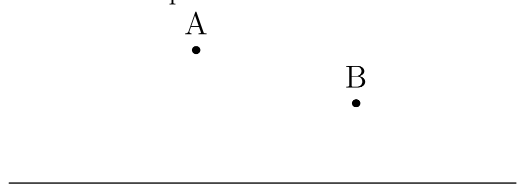
Thank You For Reading!

4 The Hardworking Farmer

We start with our third game. Actually, it's not a game, but a kind of puzzle (where people have to think.) This was the Question :-

"A farmer, living in house A, has to regularly fetch water from the nearby river and returns to a different house B. Is there a shortest path which he can use everyday so that he gets as least tired as possible?"

To make this problem a bit understandable (for them), here's a picture :-

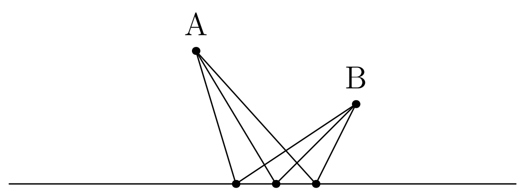


Suppose the farmer lives in House A, and the river is just a few meters away in the form of a straight line, and then he goes back and returns to House B.

(The students did not accept it to be a house, so I had to draw real houses there :P. After drawing the houses, they started drawing trees, the sky, grasses, the Sun giving colours to everything, etc. It looked like a drawing class. :-)

But eventually, I went to the problem, and decided to tell it like a story. "Suppose this farmer is very hardworking, he needs water for his crops, and everyday he has to go get this water from the river and store it in a different house. So if he can manage to find a shortest path, he will always use that path from that day, which will help him save time as well as energy."

For this problem, we would assume that we have an open ground. To show them what I mean by "a shortest path", I drew them the following paths :-



Ayesha asked, "Do we have to find the shortest path among these 3 paths?". I said, "No, this is an open ground. There are many paths from which the farmer can go from house A to house B (touching the river), I just drew 3 of these for an example."

"So, which of these paths has the least length? Is there a way to find it out? If the farmer can find this out, then for his usefulness he will use this path everyday to fetch his water.

After they thought for a few minutes, I said, "Some of you may/may not be aware of this, so I am explaining it at a brief. Suppose the farmer did not have to visit the river. Suppose he has to just reach House B from House A. Is there a shortest path for that?"

Ayesha and Sumana replied, "Yes! When it is a straight line."

"Exactly." I replied. "It is an open ground, and if you take a path which is very much curved, then definitely the length of that path is going to be large. So, the shortest distance between two such points (we can consider them houses also), will be a straight line."

(**Note**:- The real proof of this is very difficult, as it uses Variational Calculus (which even I am not aware of), so we will keep it till here. For those who needs an Intuition, we can still approximate a curve as a longer straight line, and apply Triangle Inequality there to see that it holds.)

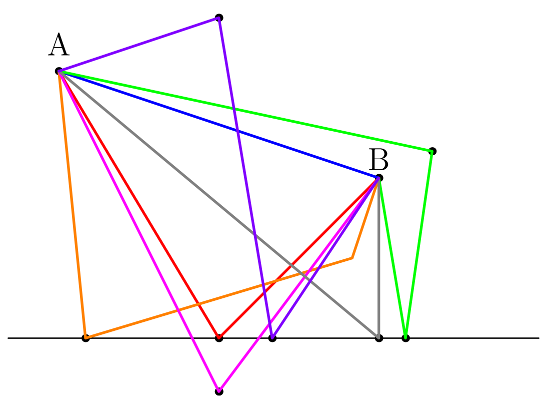
"Now we have to think a bit differently for this problem. Here the farmer also has to reach point B, but first he has to touch the river (or the line), and then reach B. What is the shortest path in this case?"

The students started drawing different paths in the jamboard (like the way I drew 3 paths before to show them), and were asking me if this/that path was the shortest path or not. I said, "All the paths seem to be very equal to each other, in length, but to find which of them is the smallest, we have to measure each of those paths. But that is very time consuming, and even there are a lot and lot (in fact, infinite) paths which exist from A to B." Everyone agreed with this.

Sumana then asked, "So the farmer must take water from the river? Can he not take it?" I said, "We already saw the case when he just goes between houses. But if he has to cross the river as well, can we find the shortest path in that case?"

Sumana then drew a path, which starts from the river, goes to A, and then to B. I reminded her that the path will start from A, (since the farmer comes from the house A), then goes to the river, and then he goes to B. I said, "All the paths will be in this sequence."

Once again, the students drew different types of paths in the jamboard, and were asking me if this/that path was the shortest or not. This figure is a summary of what they had all roughly drawn in that class.



Clearly it seems that they have understood the problem, but they are a bit confused on how to approach it. This was a very nice approach of them, they are just trying to think themselves by considering all the possibilities of paths that are possible. In fact the solution to this problem (as I will reveal it), is very non intuitive, i.e. , it's not obvious on how to find it out at first till someone sees it before. So the students were trying an usual approach, and this is what it should be done. In my opinion, for every problem at first we have to think what the problem asks, then try finding a reasonable approach to work on, and then see if it works. If not, then we move on to other approaches.

So I tried encouraging them by saying, "How do you know this/that path is the shortest? Do you have any Reason of why you think that path is the shortest? Of Course you can measure them, but in reality it will be very hard for a farmer to measure just a single path, and there are so many paths that you can't just measure all of them."

I could also make some few logical claims on comparing with a few paths, which everyone seemed to agree :-

"If you all can notice the paths, we can actually kind of guess that some paths are longer than the others.

For example, the green path takes an U-turn, so it definitely looks longer than the red path (From our experiences, we prefer going straight than taking U-turns :).

The violet path, seems to go in the up direction a bit, then goes down and touches the river, and then goes to B. So the (going up) part was surely not necessary, it could have been gone down at once. (**Exercise**: Try proving without any mathematical formulas/ideas, why the violet path is not the shortest path we are looking for.)

The pink path, interestingly enough, touches the river twice, but hey! we can also kind of see that it's a bit longer than some of the other paths. This will be a good exercise for the Readers. (**Exercise** :-
(i) Prove that the Pink Path is not the shortest path.
(ii) Further Prove that any straight Path which touches the line(or the river) twice, is not going to be the shortest path.)

So we found some really great observations with those paths, which the students drew on the board. But still those can be said as good exercises, and it is limited in a lot of ways. We just saw with straight paths, whereas we could have curved paths as well, although by Intuition we may say they are longer, but that is not a way to claim those.

As I was running short of time, I decided to discuss the general problem now. As a Final Hint, I said to them, "The farmer has to fetch water from the river, but to also find the shortest path, he has to use the river in some clever way. Have you all seen reflection on the mirrors, and on the water surfaces? We are going to use exactly that. But how are we supposed to do it? "

One student answered, "Then we should use the river somehow."

"You are right." said I. "So you can reflect through the river, but wait, you just have 2 houses, a farmer, and an open ground. You obviously can't reflect the ground, so what should we reflect?"

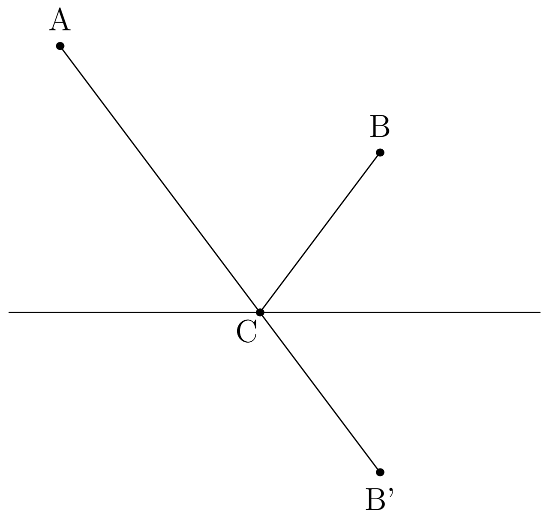
Another student answered, "Should we reflect the farmer?"

I said, "That is a good question. But the farmer seems to move at every moment, first from house A to the river, then from the river to B. So it seems unlikely that reflecting the farmer will help. So we should reflect the houses, since they just remain where they are."

And indeed, reflecting any one of the houses is going to work for our purpose. "Suppose, we reflect house B through the river, at a point which we can name house B'. (that's like B's brother, but to distinguish it we make a dash sign at the top)"

Sumana then asked, "Should we Join the points A and B' then?"

I said,”Exactly, and we can also notice that this straight line cuts the river in some point (which we can name C).



Now can anyone of you tell me what should be the shortest path we are looking for?”

One of the students replied, ”Is it the path from A to C, then C to B?”

I said, ”Exactly. That is the answer.”

Taher and Sumana said, ”I had seen this type of problem before!”.

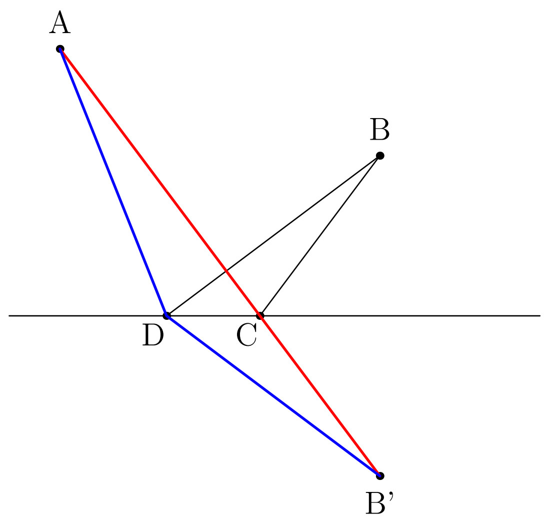
I said, ”Great! now can anyone tell me why is this path the shortest? Why not the other paths which we all drew before?”

Ayesha said, ”I think because the other paths will bend a bit when we make them join to B’, and this path is absolutely straight.”

Sumana said, ”I think in the other paths, the part of the path which follows through the reflection in the water, will be a bit behind this path, which is very near to B’.”

Sumana had the idea, but she couldn’t tell it properly. But Ayesha was absolutely right, full marks to her.

”To see why this is the shortest path, let’s just draw another path which is going from A to the river, and then to B. Let that path meet the river at some point D. Now if we join DB’ (the reflected B), we get another path. Hence, now we have two distinct paths, one is just AB’ and the other is going from A to D to B’. ”



"Now we just have a red path and a blue path. Which path is longer? It's obviously the blue path, since we already know that between two points (in this case they are A and B'), a straight line is the shortest distance!)"

And since this holds for all the other blue paths, we are finally done! The farmer now has a way to save time and save energy!!

I will end this with 2 **Exercises**.

(i) Why does the reflection work? How do you show that the paths AB and the path from A to C to B, are just the same thing? Note that we did not discuss this in the session, as it is a bit of high standard for the 4th graders.

(ii) (For Those who can understand High School Physics) Do you notice any similarity between this problem and a topic in Physics? Does this remind you of any Law? Can you now prove this law? (Just with the help of this Problem?)

That was all from our third puzzle. And this completely sums up all our discussions we had from our first Session. Overall it was really fun to have my first class, and to have my first attempt in writing a diary. I hope (to all readers) that you thoroughly enjoyed my diary. I hope to continue the making of diaries, about the other sessions which I will take with the students.

Thank you for Reading!

SOLVING NONTRIVIAL PROBLEMS

Souradip Das

April 5, 2022

Session 2

About Myself: Hello! I am Souradip Das, a student of class 9. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in our 2nd session.

1 Introduction

We had 5 students in the session. Unlike the first session, they all were 4th graders and classmates from the Filix School. Their School Teacher also took a class with us, since they had to take some time learning the new facilities in the online platform, which were new to them.

We had discussed 3 problems, but unlike the first session, where I used some of my own problems, in this session our Cheenta Academic Coordinator provided me with their own questions to be discussed on. So all these sessions from now on will be more like Problem Solving sessions.

2 Question 1

This was the First Question that we started with :-
"Today is a Sunday. Francis starts to read a book with 290 pages from today. On every Sundays he reads 25 pages and on all the other days he reads 4 pages, with no exception. How many days does it take him to read the entire book?"

(At some moments there were some small interruptions with my microphone and they were not able to hear me, so there were some problems from that part which I faced.)

So I started explaining the problem, "There was a boy named Francis who started reading a big book of 290 pages, and he starts reading it from "today", and it was also mentioned that "today" was a Sunday. It was also given that in every Sundays he would read 25 pages, and on all the other days, he would read just 4 pages, which is comparatively lesser than on Sunday. And the phrase, "with no exception" - means that Francis will follow this pattern of reading from every week."

So I opened a Jamboard, and started writing the main idea of the problem.

In that moment, Aarush asked, "So sir, we have to find how much he reads the book in 7 days?"

I replied, "No, you all have to find that how many days he will take in total, to read the entire book, along with the conditions given."

So, I started writing, "In Sundays, he reads 25 pages, and on all the other days, he reads 4 pages."

Aarush replied, "First we have to find that how much he reads in a week, and in a week he will read $(25+4)$ pages."

Aarush was almost correct in his approach, but in a week there are 7 days, not 2 days. So he wouldn't really read $(25+4)$ pages, but instead he would read $25+4(6)=49$ pages.

So here I replied, "Yes we need to find the number of pages he reads in 1 week, but will he really read 29 pages every week? There are 6 days in a week other than a Sunday, which are till Monday to Saturday.

Aarush then again said, "So he will read 29 pages in a week?"

I thought that he probably had some problems in visualising the situation. So I asked, "Why is that so? You can see that in each Sunday he will read 25 pages, and on each of the other 6 days from Monday to Saturday, he will read 4 pages each, so does that make 29 pages?"

Then Shubhadeep answered, "Sir, on every week he will read 49 pages."

Finally, Shubhadeep was right. We also had to remember that in the problem, it was given that Francis started reading the book from a Sunday, and from the every other day he reads 6 books."

The main thing to note here was that, this thing was periodic, i.e. this process will continue like this on and on.

Once, Subranil asked, "Sir, we have to multiply 25 with 7 , because I think he is reading 25 pages every day."

I replied no to him and said that, "It is not that he is reading 25 pages everyday, he is reading 25 pages every Sunday, and on every other days he just reads 4 pages." So at this point, they all understood why the person is reading 49 pages every week.

Then Aarush replied, "Then we can divide 290 by 49, and get that how many weeks in total are needed."

I said, "Yes, very correct, but then after division the quotient would be 5, and the remainder would be 45. Does that give anything?"

(It was at this moment that I was having many problems with my microphone, so finally I had to join from a new device, and it took 10 minutes extra. In that moment the students were asked to think on their own about the problem.)

So when I just returned, Aarush gave the final explanation to the problem. He said, "So for 45 remaining pages as the remainder, in 1 Sunday he can read 25 pages, and on 5 other days after that Sunday, he can in total read $4(5)=20$ pages and that in total makes 45 ways. So in 56 days he can read the whole book. "

Aarush was totally correct, everything was done, till at the last part which I think was, that he made a calculation error.

So I fully completed explaining the problem this way. At first, in 5 weeks, 245 pages are already done. Now 45 pages are left. Now again from these 45 pages, a new week starts and Francis again starts reading from a Sunday, and from these week he starts reading 25 pages, then he reads 4 pages from every other day.

How many more extra days will he take? He will take 5 more days apart from a Sunday, to completely finish the book, since $25 + 4(5) = 45$ pages are done.

So, overall, Francis will take 5 weeks + 6 days, or 41 days, to fully complete reading the book.

3 Question 2

Then we started with our Second Question.

"The number 35 has the property that it can be divided by its unit digit, because 35 divided by 5 is exactly 7 . The number 38 does not have this property. How many numbers bigger than 21 , but smaller than 30 have this property?"

This question here was a bit interesting, as it focuses on a specific property followed by some numbers.

So I started with the statement, "The number 35 has the property that it can be divided by its unit digit, because 35 divided by 5 is exactly 7". What exactly does this statement mean?

I tried explaining to them, that if you divide the number 35 with its unit digit, which is 5, then there will be no remainder. Now I asked anyone to volunteer why the number 38, as mentioned, does not have this property.

Smita volunteered, "Is it because 38 cannot be divided by 5?"

I replied, "No!, first look at the units digit of 38. The units digit of 35 is 5, but what is the units digit of 38?"

It was at this point that I understood that the students were not familiar with the definition of "an units digit". So I told them that the units digit of a number, is just the last digit of that number, however big/small the number is. I took some examples of unit digits, and at this point they understood what the unit digit of a number is.

So, back to the question, "What will be the unit digit of 38?"

Smita said, "It should be 8, and on dividing 38 by 8, there will be remainder. So it cannot be divided."

I replied, "Exactly, and so that is the definition. Now the question says, that how many numbers bigger than 21, but smaller than 30, have this property?"

In other words, I explained that how many numbers in this group from 21 to 30 are there, such that when we divide that number by its unit digit, there will be no remainder?

Now it was just a small Trial Error from all the values $\{22, 23, \ldots, 29\}$ and once they figured out how to approach this, it just took them some minutes to find out that the numbers that work will be $\{22, 24, 25\}$ and so the answer will be 3.

4 Question 3

Then we started with our third question.

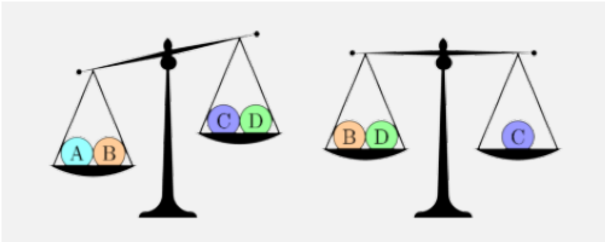


Figure 1: Weighing Balances with Balls

"The figure above shows 2 different weighing balances filled with 4 different types of balls. If each of the balls weigh either 10 or 20 or 30 or 40 grams, which ball will weigh 30 grams?"

At first I asked all the students if they had seen a weighing machine, and how it actually works. Everyone more or less, had seen it before, especially at the markets. So it wasn't difficult for me to explain them the situations which are given by the weighing machines.

They also did not know algebra, nor the usage of variables. So this problem was great to show some of the basic concepts of using variables, which were also very convincing because of the weighing balances.

So, as I started, from the figures (taking the 2nd weight), all of them agreed that we can take

$$(B + D) = C$$

in this way. Just because on the 2nd balance, the masses of B and D together equals C, and so the balance is upright.

Aarush then asked, "Then shouldn't we get that B must weigh 10 gms, D must weigh 20 gms, and C weigh 30 gms?"

I replied, "No, not necessarily. It might also happen that B weighs 10 gms, D weighs 30 gms, so C weighs 40 gms. We just cannot guess the weights of the balls, we need to find them ourselves."

Similarly, from the 1st weight, since it was tilted to the left side, we can take

$$(A + B) > (C + D)$$

Just because the weights of A and B together is heavier than C and D, and so it is tilted to that side.

So, from $(A + B) > (C + D)$, I claimed that we can take :-

$$(A + B) > (B + D) + D$$

Everyone agreed to this, once I pointed out that I just put C as $(B+D)$, because we just knew before that C equals $(B + D)$.

Then, I asked them if I can cancel B both sides to get :-

$$A > (D + D)$$

Some students were not fully agreeing to this, so I took some examples to show them why it's true.

"If you have a large number and a small number, and if you compare them, then the large number will be larger. But now here if we add anything equal in both sides, will anything change?"

"Suppose we have $5 > 2$. Again, $(5 + 1) > (2 + 1)$ and again $(5 + 100) > (2 + 100)$. So whatever we add to both sides, this $>$ sign will never change."

So hopefully they understood till here, and so we can cancel B both sides to get :-

$$A > 2D$$

Aarush replied, "This problem is too tough."

Since the students did not have prior knowledge of algebra before, I think that this was the reason why the problem was looking tough for them. So I agreed to him and told him that we will do this Problem Step by Step now.

Now comes the most interesting part of the problem. Apart from the Algebra, instead of guessing the weights of each of the balls, we did a systematic Casework on the possible weights, and with the info we have figured out.

So I started, "The weights of the 4 balls can only be either $10, 20, 30$ or 40, nothing other than that. So for now, just assume that A weighs 10, but then what is the weight of D?"

Nobody had an answer to that question, preferably because there was no possible weight at all. So I said, "D can have no weight, because no possible weight of D satisfies the equation $10 > 2D$." Everyone agreed to this.

"So, the weight of A can never be 10. So, can the weight of A be 20?"

At this point, Smita said that she couldn't understand the question from the start. I tried explaining to her, how we got the two equations $(B + D) = C$ and $(A + B) > (C + D)$, from the weighing machines, and how the machines actually work. And after that (hopefully) she claimed that she understood everything we did after that.

So I resumed, "So, can $A's$ weight be 20? We all found out that A cannot weigh 10. So can A weigh 20?"

"For now, again assume that A weighs 20. Now all the possible weights of D can be just $\{10, 20, 30, 40\}$. If D was 40, then this was obviously wrong. We would get $20 > 80$, very wrong. Similarly if D was 30, again it would be wrong. If D was 20, again wrong. But what if D was 10?"

Smitha said, "If D was 10, we would get $20 > 20$, which is again wrong too. So D's weight can be nothing." This was very correct.

I replied, "So this just gives us, that A's weight cannot be 20. So you all can see how we deduced something very interesting from here."

Now, I came back to another case, "Can A's weight be 30? Yes, now it is possible. We can take D's weight to be 10, so it follows $30 > 20$. Also notice that this is the only possible weight for D, if A weights 30."

"Now the weights of B and C are not known to us, so we can assume some of their weights.

(i) If B would have a weight of 10, then to satisfy $(B + D) = C$, C would have a weight of 20. This works, and when we put these weights in the weighing balances, they satisfy what was being said. So $(A, B, C, D) = (30, 10, 20, 10)$ is a solution.

(ii) If B would have a weight of 20, then to satisfy $(B + D) = C$, C would have a weight of 30. This also works, similarly, and we have another solution $(A, B, C, D) = (30, 20, 30, 10)$.

(iii) If B would have a weight of 30, then to satisfy $(B + D) = C$, C would have a weight of 40. This also works, similarly, and we have another solution $(A, B, C, D) = (30, 30, 40, 10)$.

(iv) Now, if B would have a weight of 40, then to satisfy $(B + D) = C$, C would have a weight of 50, which is not possible. So this does not work.

Now I asked, "Have we found all the solutions? Or are there still some solutions left?"

Smita said, "I think we have checked all the cases."

I reminded them that these cases are when we assumed A to be of 30 grams, but A can weigh 40 grams too, so we did not check that case yet.

So I said, "Assume A now weighs 40 grams. Again here, D can only weigh 10 grams, which satisfies $A > 2D$. If D was of weight 20 grams, then $A > 2D \rightarrow 40 > 40$, which is wrong, and so on. So there is again just 1 choice of D, that is, D has to weight 10 grams."

"Once again, we do not know the weights of B and C. So we can check cases on B's weight."

(i) If B weights 10, C weighs 20, and we have a solution $(A, B, C, D) = (40, 10, 20, 10)$.

(ii) If B weights 20, C weighs 30, and we have a solution $(A, B, C, D) = (40, 20, 30, 10)$.

(iii) If B weights 30, C weighs 40, and we have a solution $(A, B, C, D) = (40, 30, 40, 10)$.

(iv) If B weights 40, then C would weigh 50, which is a contradiction. So there are no other solutions here.

"So, we total get $\underline{6}$ solutions, and this fully completes the problem. Notice we have checked all the cases, and no more case is left."

(**Note**:- In the original question, it was to be assumed that each of the 4 balls would have different weights. That would have given only a single solution to this question. [That case is when $(A, B, C, D) = (40, 20, 30, 10)$]. I failed to understand that in the class, and so I figured out all the possible solutions, along with the intended solution, in the class itself, and discussed that instead with the students).

So that was all which we had roughly discussed in the 2nd session. In total there were 6 problems supposed to be discussed, but I could discuss only 3 problems with them due to time shortage. Hopefully I was able to explain everything necessary for the problems, and overall it was a great session.

Thank you for Reading!

A CLASS ON PATTERNS AND PUZZLES

Souradip Das

April 5, 2022

Session 3

About Myself: Hello! I am Souradip Das, a student of class 9. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in the 3rd session.

1 Introduction

We had 5 students in the session. They all were 4th graders and classmates from the same rural primary school. Their School Teacher also took a class with us, since they had to take some time learning the new facilities in the online platform, which were new to them.

We had discussed 4 problems (and 1 problem partially), but unlike the first session, where I used some of my own problems, once again, in this session our Cheenta Academic Coordinator provided me with their own questions to be discussed on. So all these sessions from now on will be more like Problem Solving sessions.

2 Question 1

This was the First Question that we started with :-
"Grandma's watch has an hour, minute and a second hand. We don't know which hand does which job, but we know that the watch tells the correct time. At 12 : 55 : 30 hours the watch looked as pictured."

"How will the watch clock at 8 : 11 : 00 hours?"

Before starting this problem, I asked everyone if they have seen an analog clock in their houses. Everyone did, and so they already know the mechanisms of how to figure out some specific time.

So they had no such problem in understanding the question itself. There could have been 1 problem which I said, "Now in the question it was said that the clock shows that time, on 12 : 55 : 30 hours. This notation might be a bit different, as in usual we have times like 12 : 55 PM, but can anyone of you tell, from where is this 30 coming?"

Arqam(and everyone else) immediately replied, "It is just denoting the no. of seconds."

I said, "Exactly, so 12 denotes the number of hours, 55 denotes the number of minutes, and 30 denotes the number of seconds. So this clock shows 12 : 55 : 30 hours. And we have to find what type of clock will show 8 : 11 : 00 hours."

There was also a small advantage, the number 8 : 11 : 00 hours just shows that no seconds have passed, so here we can consider it to be just a standard clock showing the time 8 : 11, or 8 hours 11 minutes.

Now I said, "So we have an hour hand, a minute hand, and a second hand. We just have to figure out which hand is what, and then we will mark 8 : 11 in the original clock, and we would be done."

At this point I understood that they did not actually know on how to make sketches on clocks, if a specific time was given.

So I tried explaining it to them, "The hour hand will be slightly at a higher position than 8, because it has already gone some minutes past 8, and now the minute hand moves a twelfth of a circle, on every 5 minutes, so after 11 minutes it will be just a bit near 2, but already past it. And since the second hand has not moved at all, it will stay at exactly 12." Hopefully everyone was able to understand it, by looking it from the picture.

So, to plot 8 : 11 on a clock, it would look something like this :-

Now the part left was that in the original picture, there were 3 hands of 3 different sizes. Suppose I call them the short hand, the medium hand, and the long hand, and these match in some order, to the actual hour hand, the minute hand, and the second hand.

In the original picture, I asked the students if they can figure out what would be the hour hand.

Almost everyone said, "It would be the hand with the smallest size." So the students got confused with the usual shape of the hour hand. I claimed, "But if that would be the hour hand, then it would have only reached 6 hours, whereas the time shows 12 hours, so that cannot be the hour hand. So what can the hour hand be?"

Sumana asked, "Then is it the medium sized hand, which is the hour hand?"

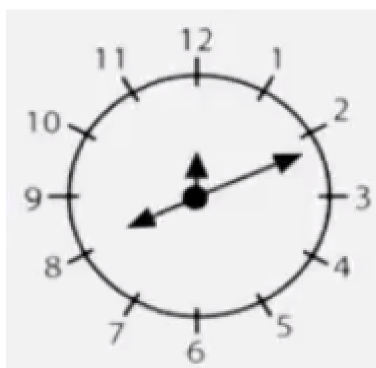
I said, "Yes, exactly. Can you tell why? If the long hand was the hour hand, then again it would have reached just 11 hours, but the medium sized hand has travelled 12 hours at first, after a full rotation, then again almost another 1 hour, which works because the time $12:55:30$ is very close to 1'o clock, and that hand is very close to 1."

"So we got the hour hand, and what would be the minute hand?"

2 to 3 students replied together, "The long hand will be the minute hand, and the short hand will be the second hand."

I said, "Yes, correct. the long hand will be at 11 because it has exactly covered $11(5) = 55$ minutes, and the second hand will be at 6, because at this point, it has exactly covered half a minute, or 30 seconds."

So, we just drew a new sketch, but with the desired changes of the sizes of the hands. The new clock would look like this.



So, this was the final answer.

3 Question 2

So then we started with our second question.

"In an arithmetic sudoku, the values $1, 2, 3, 4$ appear exactly once in each row and each column. Which value belongs in the grey square?"

At first, I asked everyone if they were familiar with the "sudoku" game, and if they have played it before or not. Some students were not familiar with the game, so I explained it to them how to play it.

1 × 1		1 × 3	
2 × 2	6 − 3		6 − 5
4 − 1	1 + 3	8 − 7	
9 − 7	2 − 1		

To play sudoku, at first we take a $4 * 4$ board (it can be played for any $n * n$ board, for n being a natural number, and $n > 3$ for convenience), then in each square we either put a number from $1, 2, 3, 4$, such that every row, every column, and every cornered $2 * 2$ box will contain the set $\{1, 2, 3, 4\}$. If this can be achieved, then the sudoku will be complete.

Some of the students were still not understanding how to play it, so I took some examples of games and showed them what exactly was to be done. After that they (hopefully) understood the game.

I tried explaining it to them like this, "Suppose you have a $4 * 4$ grid. Now, in every row, in every column, and in every cornered $2 * 2$ square, try filling them with just $1, 2, 3, 4$'s such that in every such row/column/square, no 2 of the same numbers are equal.

To show them an example, I took a column, and filled the first 3 squares with $1, 2, 3$ in the order. Then I asked them what should be the number in the 4th square."

Sumana answered, "It should be 4".

I replied, "Correct, because then only this column will have 4 different numbers. We need to put the values, just like this-, to all the columns, rows, and the $2 * 2$ squares. If you can put the numbers in these way, then only the sudoku will be complete."

I drew them an example of a sudoku, which satisfies the conditions. Then I said, "Note that this sudoku is such that, each of its rows have different numbers, each of its columns have different numbers, and so do the $2 * 2$ squares have different numbers. So this works."

Hopefully then they understood on how exactly to play a game of sudoku. So we came back to the problem.

The original problem had squares already filled with numbers involving operators. So, at first with basic addition, subtraction, and multiplication, we fill up all the known numbers in the boxes. Then we get this grid.

1		3	
4	3		1
3	4	1	
2	1		

After that it was just a matter of minutes that the students were solving the sudoku, and Sujata first answered, that the answer will be 3.

And then, with simple logic (which everyone agreed), I solved the sudoku on the board, and the answer indeed came out to be 3.

Here is the completed sudoku which I had done during the class on board :-

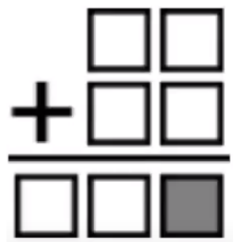
1	2	3	4
4	3	2	1
3	4	1	2
2	1	4	3

4 Question 3

Then we started discussing the third problem.

"In each box exactly one of the digits $0, 1, 2, 3, 4, 5, 6$ is to be written. Each digit will only be used once. Which digit has to be written in the grey box so that the sum is correct?"

(**Note**:- For convenience, we assumed that 0 cannot be placed as the leading digit of any number.)



So at first I told the students to read the question, and then I tried explaining the problem to them.

"In these 7 boxes which you can see, we need to fill up each of these boxes with 7 different digits $\{0, 1, 2, 3, 4, 5, 6\}$, and we just have to fill them such that the addition shown will be correct. If this can be solved, then we just have to find out the possible number in the grey box."

The question was not that hard to be understood, so I gave them some time (like 5 minutes) to think about the question, and in that moment I was myself thinking of an approach to do it.

At some moments I volunteered to start discussing about the problem, but some of the students wanted some more time to do it, so I stopped.

After some time, Tutul said, "I think I have a solution. $62 + 43 = 105$ works, with all the digits just used once."

Sujata also then said, "I also saw that $42 + 63 = 105$ also works."

I said, "Wow, very nice. This has been solved. Can you all tell how you did it?"

Arqam said, "I think it just happened by guessing." I said, "Yes, a bit of guessing is needed, but is there a good way to get the answer?"

It is at this moment, that I start taking cases. "Suppose I start taking classes with all the possible places where I can put the number 6 into."

"Suppose, at the units place, I take $(6 + 4) = 0$, then the problem which we will face is, even if we put the next 2 biggest numbers 5 and 3 in the tens place, we will never get a three digit number."

"Notice, by the same reason, in the units place we can't do $(6 + 5) = 1$ too, because we again won't get a 3 digit number."

"This just shows, that I can't put a 6 in the units place in the added numbers, as there is no way to get a solution like that."

"Suppose now I put 6 at the tens place in the added numbers. Now note that we must have a 4 or a 5 as the other digit, to get a carry over."

(i) If it was $(6+4) = 0$, then we would have a situation of $\overline{6x} + \overline{4y} = \overline{10z}$, and this gets solved if we put $62 + 43 = 105$, or $63 + 42 = 105$, both of which works.

(ii) If it was $(6+5) = 1$, then we would have a situation of $\overline{6x} + \overline{4y} = \overline{12z}$, and this is impossible as there is no way to get a carry over at the units place.

By now, the students were hopefully able to understand these as I was showing these cases one by one in the boxes.

Now, suppose 6 was at the hundred's place. Then the case would just be the sum of two 2 digit numbers, equalling a number with 6 at the hundred's place. But the maximum sum of two 2 digit numbers can be just $99+99 = 198$, which is even less than 200. So this is obviously impossible.

"Now we still have 2 more remaining cases. We will check each of them now."

It was at this point that Arqam suggested to give another problem, as it was probably too hard. I told him that we have almost done it and we will complete discussing it soon.

(i) If we had $\overline{ax} + \overline{by} = \overline{m6n}$, then at the ten's place, we can either have a sum $(1+5)$ or $(2+4)$. Notice that there is no point of carry over, as in the units place the maximum sum can just be $(5+4)$, not exceeding 9.

(a) If it was $(1+5)$, then $\overline{1x} + \overline{5y} = \overline{m6n}$. Again there is no carry over, so the hundreds digit must be 0, (although this is impossible), but then we will have digits $(2, 3, 4)$ remaining at the units place, and they cannot be filled.

(b) If it was $(2+4)$, then $\overline{2x} + \overline{4y} = \overline{m6n}$. There is no carry over, so the hundreds digit must be 0 (again impossible), but then we will have digits $(1, 3, 5)$ remaining at the units place, and they cannot be filled.

(ii) If we had $\overline{ax} + \overline{by} = \overline{mn6}$, then at the unit's place, we can either have a sum $(1+5)$ or $(2+4)$. Again there is no way of carry over, as we cannot get a sum of 16 to do a carry over.

This case was handled similarly like the cases above, there was no difference, and again there will be no solutions.

And there are no other ways to put a 6 here, so we are completely done!!

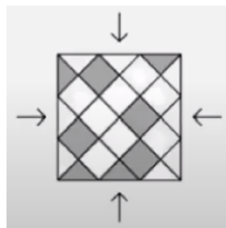
So, the only solutions will be $62+43 = 105$, or $63+42 = 105$, or $42+63 = 105$, or $43+62 = 105$. And in all the cases, the number in the grey box will be $\underline{5}$, which is our answer.

(**Note**:- All of these was not done that much rigorously in the class, but it was intuitive and so it wasn't that hard to get the logic of all the solutions. And 1 solution was anyways enough to convince them to get the answer.)

5 Question 4

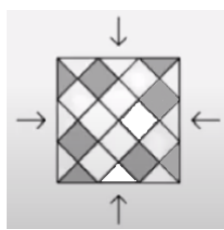
We then began with the 4th question.

"A square floor is made up of triangular and square tiles of grey and white colour. What is the smallest number of grey tiles that have to swapped with white tiles, so that the floor looks the same from all the four given viewing directions?"



This was one problem, which I misread and so even I wasn't able to solve it properly with the students, so we had a different discussion other than the intended solution.

Intended Solution :- Since we have to swap grey tiles and white tiles, to get the same view from all the 4 directions, the figure should have rotational symmetry of order 4. We can achieve it like this :-



And this can be achieved just by swapping 1 grey triangle, and 1 grey square, so the answer is 2. The main idea was of rotational symmetry, that rotating the paper in all the 4 directions, won't change its look.

In the class we had done a different discussion about the problem.

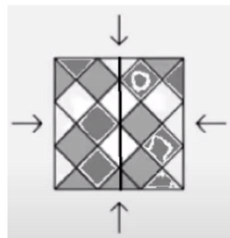
Discussion:- At first when we started the problem, I misinterpreted it, and concluded that we need to find that how many grey pieces need to be coloured white, such that it looks same from all the 4 positions.

The original solution, wanted us to swap grey pieces with white pieces, but I considered them as colouring the minimum number of grey pieces into white pieces.

Then, I again misinterpreted the problem.

I had the idea that we need to allow the figure to have rotational symmetry of order 4, but this might not be explainable to the students. So I tried showing them some idea by reflection, and then I also thought that the arrows were meant for reflection only. (but these were 2 different things, and here I misinterpreted the problem).

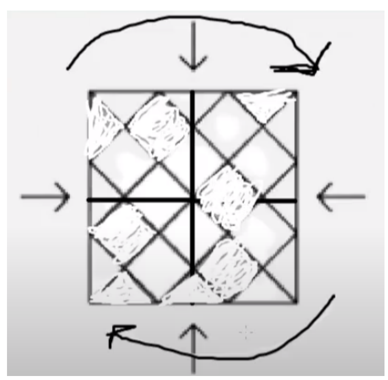
So, if we try folding the pattern midway vertically like this :-



Then for it to be symmetric, it should look like this. (With the new grey squares)

So, this was the discussion that happened in the class. "What do the arrows actually tell us? It just tells us, that if we fold the paper in 2 ways, vertically or horizontally, then it should match the grey squares. (This was completely wrong, as it was on reflection, and it won't be used here.)

So, on coming back to the problem, with 2 wrong ideas, (one on the reflection, and the other on colouring the pieces white, instead of swapping grey and white pieces), the final conclusion which the students were able to figure out, was this :-



The conclusion was that, all the squares of the pattern should be white. Now this obviously looked wrong, and so at that moment I thought if I was making some mistake.

It was at this point, that I realised that the idea of reflection does not really work (I still couldn't realise that the tiles were supposed to be swapped).

At this point Sumana said , "Sir, I am not able to understand anything."

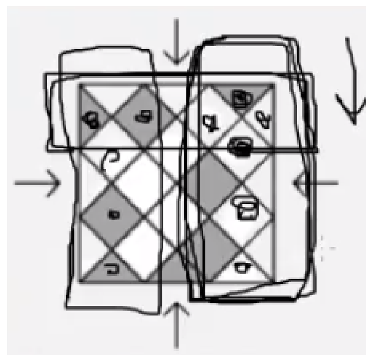
Yes, so I replied to them that I will try explaining the problem again, from the start. I opened a new jamboard for the figure, and started with the problem.

I said, "The problem cannot be solved like that folding way before. It needs to be solved differently. Now, just imagine that my eyes of view are on the 4 directions, up, down, left, right."

"Consider the 1st column, which originally has 1 black triangle, 1 black square, 1 white square, 1 white triangle. Suppose this is my view when I am looking at the top."

"Now suppose I change to the left side view. Then the same pattern should be there, but instead the order should be from the first row."

"So this pattern needs to be followed from the down point of view, as well as from the right point of view."



But at this point I still didn't get the fact,that the tiles of opposite colour were to be swapped, and as I didn't get any pattern, the same problem remained.

So, as I was with the idea of colouring grey pieces white instead, again the same unexpected result came. This point of view argument will work, only if all the tiles become white.

So at this point (due to short amount of time), I had to leave the problem and end the class. This problem was the only one which I misread and myself got it wrong, and so this remained unsolved.

So that was all which we had roughly discussed in the 3rd session. In total there were 6 problems supposed to be discussed, but I could discuss only 3 problems, (and 1 problem partially) with them due to time shortage, with 1 problem also being incorrectly solved by me. Hopefully I was able to explain everything necessary for the problems (other than my last mistake), and overall it was a great session.

Thank You for Reading!

SOLVING 2 INTERESTING PROBLEMS

Souradip Das

April 15, 2022

Session 4

About Myself: Hello! I am Souradip Das, a student of class 9. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in the 4th session.

1 Introduction

We had 2 students in the session. They were 4th graders and classmates from the same rural primary school. Their School Teacher also took a class with us.

We had discussed 2 problems, but unlike the first session, where I used some of my own problems, once again, in this session our Cheenta Academic Coordinator provided me with their own questions to be discussed on. So all these sessions from now on will be more like Problem Solving sessions.

2 Question 1

We started with our 1st question :-
"Fill in the boxes with numbers from 1 to 5 to get the correct equality (each number will be used exactly once):-

$$\Box + \Box = \Box \cdot (\Box - \Box).$$

It suffices to give 1 example."

At first, I tried explaining the problem to the students. "So there are 5 different boxes, and we have to fill all the boxes with the digits $\{1, 2, 3, 4, 5\}$ in some order, such that each digit is used just once, and this equation holds true. Finding any 1 of such solutions will complete the problem."

If for any confusion, I clarified that the equation shows the sum of 2 digits, equals the product of 1 digit, and the total difference of the last 2 digits. So, the dot is the multiplication sign.

It was also at this point that I came to know that the students were not yet familiar with negative integers, so I avoided those cases where a negative number might occur, and I gave an example.

"Suppose I fill up the digits $\{5, 4, 3, 2, 1\}$ in this order. Then we would get

$$\rightarrow 5 + 4 = 3(2 - 1)$$

$$\rightarrow 9 = 3$$

Which is wrong.

So we just have to find a solution, where the solution will be correct.
I gave like 5 minutes to let them try the problem, and after that Shubhadeep said that he wasn't getting any solutions.

Then I gave 1 solution which I found while calculating, it was that :-

$$\rightarrow 1 + 5 = 3(4 - 2)$$

$$\rightarrow 6 = 6$$

Which works.

So we found 1 solution, and this was enough to complete the problem. But this wasn't rigorous, and there were more solutions which weren't found in the class. So I also tried explaining a rigorous way to approach the solutions, but my approach was an ineffective casework on the last 2 digits which undergo subtraction, and (with the confusion of negative integers) it was getting hard for them to understand the long casework, so I left it and moved on to my next question.

Note:- There is an easier casework which is effective in finding all the solutions. (Which I didn't notice in the class itself). So, here is an exercise left to the reader.

Exercise:- Find all the solutions that work.

3 Question 2

This was our 2nd question which we discussed :-
"Six gnomes are seated at a round table. It is known that exactly two gnomes always say the truth, and they sit side by side. In addition, exactly two gnomes always lie, and they also sit next to each other.The remaining two gnomes can both lie and tell the truth, and they do not sit next to each other. A treasure seeker walks around the table and asks the dwarves where they hid the gold. Where did the dwarves hide the gold? The answer must be justified."

- The first dwarf said that the treasure was in a cave.
- The second said - at the bottom of the lake.
- The third said - in the castle.
- The fourth said - in a fairy forest.

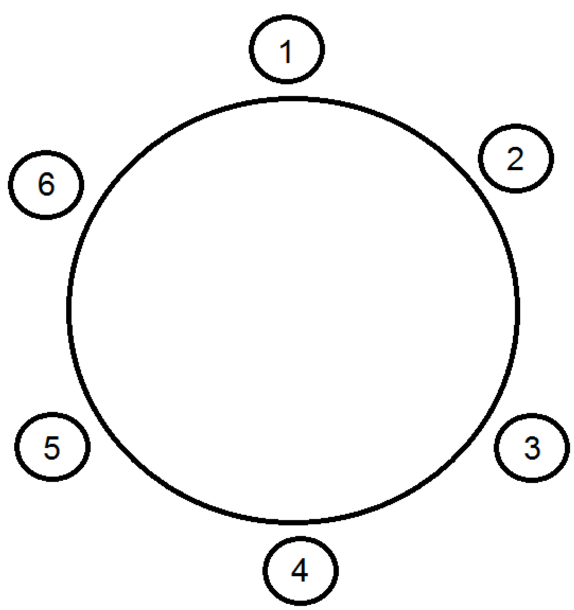
- The fifth said - at the bottom of the lake.

So basically there were 6 humans (for convenience we can consider them as humans rather than gnomes) seated around a round table. For example, I am a treasure hunter, and I came to question the people where the treasure was hidden. Everyone of them answered different statements as shown above, and the 6th person stayed silent.

Now, it was given that exactly 2 of them are truth tellers, those who always tell the truth, and they sit next to each other. Exactly 2 of them are lie tellers, those who always tell the lie, and they also sit next to each other. And the remaining 2 persons can tell the truth, or the lie, randomly, but they do not sit next to each other.

So, from here, is it possible to find the location of the treasure?

So, for convenience I drew a picture showing 6 people seated around a circle, labelled from 1 to 6.



Before starting the problem, an important thing to note here was that, 5 of the people said 5 different statements, where person 2 and person 5 said the same statement. Now, every 2 different statements cannot be true together, as the treasure cannot lie on 2 different places simultaneously. This will be used here.

Now, for now, suppose I assumed that the treasure was in a cave. So, 1 would be telling the truth, $2, 3, 4, 5$ would all be telling lies.

Now, we know that there are exactly 2 people who always tell the truth.

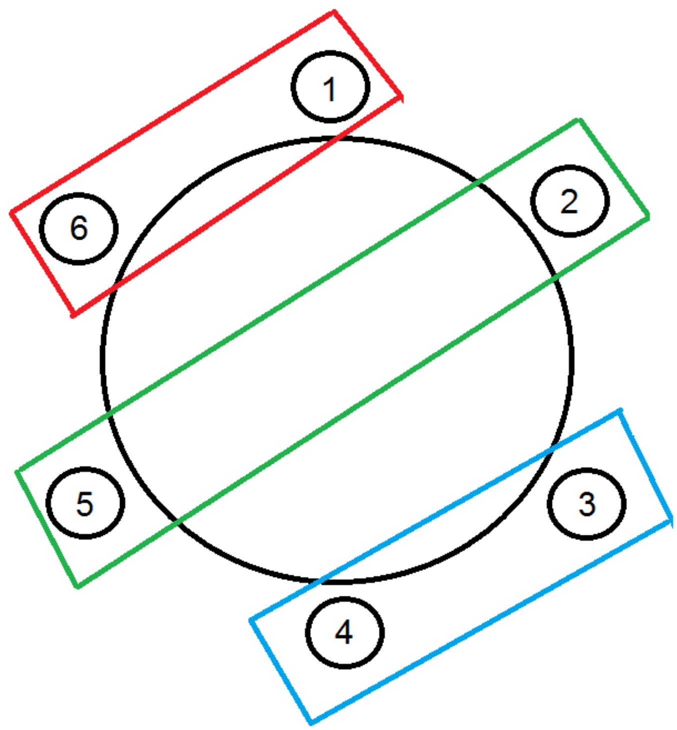
But since we already have 4 people telling at least a lie, that means the other 2 people must be truth tellers. Hence both 1 and 6 must be truth tellers, and they also sit together, so it holds.

Now, one of the students asked, "But, 6 did not tell any statement. So how can he be a truth teller?"

I said, "Yes, 6 remained silent. But a truth teller can remain silent too, so we can surely say he can be a truth teller. It's not necessary for a person to say the truth, if the person is a truth teller, he can remain silent as well."

So, among $2, 3, 4, 5$, two of them always lies, and two of them sometimes lie, and sometimes tell the truth randomly. Now their seating positions would be that, the first two people will be together, and the last two people cannot sit together.

So, we can take $3, 4$ as the people who only lie, and 2 and 5 are the people who lie, or tell the truth, randomly. So this is the situation we have :-



The first 2 people are the truth tellers, and they are seating together. The next 2 people are the mixed-type (truth tellers as well as liers), and they are not seating together. And the last 2 people are the liers, and they are seating together.

So, all the conditions are fulfilled, so this clearly is a solution.

So, since the treasure can be in just 1 place, the treasure must be in a cave.

Now, the proof is still incomplete as next we needed to show that the treasure cannot lie in some other place (that is, if the other people like $2, 3, 4$ or 5 were telling the truth, then we would arrive at a dead end). But there were 3 more cases to be checked and since I was short of time, I could only conclude that the treasure was in the cave, which was also actually enough for the justification of the answer which was needed.

But for more a rigorous solution, it's also important to do the last 3 cases too. So here is another exercise left to the reader.

Exercise :- Show that the treasure does not lie :-

1. In the bottom of the lake.

2. In the castle.

3. In a fairy forest.

Final Part:- Hence show that the treasure must lie in a cave.

That was all which we had discussed in the class. Since there were less number of students in this class, it was not that much interactive. But still I had tried explaining the problems as much I was able to do, and I hopefully made the class enjoyable.

Thank You for Reading!

CREATIVE PROBLEM SOLVING

Souradip Das

April 16, 2022

Session 5

About Myself: Hello! I am Souradip Das, a student of class 9. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in the 5th session.

1 Introduction

We had 5 students in the session. They all were 4th graders and classmates from the same rural primary school. Their School Teacher also took a class with us.

We had discussed 3 problems, but unlike the first session, where I used some of my own problems, once again, in this session our Cheenta Academic Coordinator provided me with their own questions to be discussed on. So all these sessions from now on will be more like Problem Solving sessions.

The Problem Set was exactly the same set as we had done in Session 4, but this time I did a discussion with a different group of students.
Note:- It is recommended to take a look at the previous Diary on Session 4, before reading this diary, as 2 of those problems were exactly the same problems I had discussed in the previous session, but with a different group of students.

2 Question 1

This was the first question we discussed :-

"Fill in the boxes with numbers from 1 to 5 to get the correct equality (each number will be used exactly once):-

$$\Box + \Box = \Box \cdot (\Box - \Box).$$

It suffices to give one example."
So, there were 5 different boxes, and each box needs to be filled with an ordering of the numbers $\{1, 2, 3, 4, 5\}$ such that this equation becomes true.

Like in the previous session, the students did not know about the concept of negative integers, so I avoided all those cases where if any negative integer occurs or not (and it wasn't necessary either).

I didn't have to explain the problem that much, as their School Teacher explained that "We needed to find some ordering of the numbers from 1 to 5, such that putting them in the boxes will make this equation true." Hopefully they understood this thing.

So, I gave like 5 minutes to them to try out the problem themselves.

And I said, "If anyone of you get any solution/idea to the problem, then please do inform me."

At this point, Sumana requested, "Sir could you please explain the problem again?"

I replied, "Yes, so in all these blank boxes which are there, first we take any ordering from 1 to 5. For example, suppose it is $\{1, 5, 4, 3, 2\}$. Then if we put these numbers in the boxes, we would get :-

$$\rightarrow 1 + 5 = 4(3 - 2)$$

$$\rightarrow 6 = 4$$

Which is clearly wrong, because 6 does not equal 4. So we have to find such a kind of ordering, such that this equality holds true."

Sumana claimed that then she was able to understand the problem.

Then, another student, Jesmine joined the class, and so I had to explain the problem one more time to her as well.

After that I waited for some time, to let the students try the problem.

After around 8 minutes, Neha answered, "Sir, I think I have a solution. Could you please check? I think this set works, $\{2, 3, 5, 0, \ldots$" "But, 0, cannot be used." I replied. "The only numbers that can be used are $1, 2, 3, 4, 5$."

At this point, Arqam also joined the class, so I again had to explain the problem to him.

Then Sumana said, "Sir, I think I have a solution, could you check? We put $\{2, 3, 5, ..\}$", then she asked if the last operator was a minus sign.

I replied Yes. Then she claimed that there was no solution here.

Neha had a second try. "We have $1 + 2 = 3 \ldots$" , and then she paused for some time.

I said, "You are almost done. You have used the digits $1, 2, 3$, and now you have to use 4 and 5 to complete it." It was almost done already, because we can just do $(5 - 4) = 1$ and get the answer. So :-

$$\rightarrow 1 + 2 = 3(5 - 4)$$

$$\rightarrow 3 = 3$$

This clearly works. Everyone agreed to this. And we already found 1 solution, so I agreed that we were done. I also showed them another solution which I had found in Session 4 :-

$$\rightarrow 1 + 5 = 3(4 - 2)$$

$$\rightarrow 6 = 6$$

After that, we proceeded to the next problem.
Exercise:- There are more than 2 solutions to this problem. Find a way to get all the solutions. (Also discussed in the previous diary)

3 Question 2

Next, we started with the game of Tower of Hanoi. It is a fairly popular game (single player type), but many people may not know how to play it. So these are the procedures :-
The game starts with n different sized disks (for convenience, we start with $n \geq 3$) and 3 towers, with all the n disks kept on top of one another on a single tower, and the disks are so placed on the ascending order of their sizes, from the top to the bottom.

In every move, we can move just a single disk from one tower to another, following 2 conditions :-

(i) In a tower, if there are k disks, only the topmost disk can be moved first, before moving the bottom-most disks.

(ii) A large disk cannot be placed on top of a disk, smaller than the large disk.

The objective is to rearrange the position of the disks in the exact same way, but on a different tower, but by moving the disks one by one in such a way, that they satisfy the above conditions.

And the main takeaway of this question, is that to find the minimum no. of moves needed to do the rearrangement.

(This game can be played here :- Tower of Hanoi)

And hopefully I was able to make them understand the game, and so we proceeded to play it.

The original question asked us to play this game with 4 disks in hand. So we started playing it.

It was a very interactive part when we were putting the disks from one tower to another (we were playing the game from the link given above). Everyone was interacting one at a time, to tell which disk to put in what tower, and overall it was a great interactive session.

We finally completed our task, in 17 moves. But (from the website itself), the minimum moves which we could have taken, was just 15, so I gave them the task of finding the solution in just 15 moves, and then I moved on to the third question.

(**Exercise**:- On experimenting some more on this game, with different values of n, can you guess what will be the minimum no. of moves required to complete the game, if we start with n disks? Can you prove it?

Hint:- Use Recursion. (It's an advanced topic))

4 Question 3

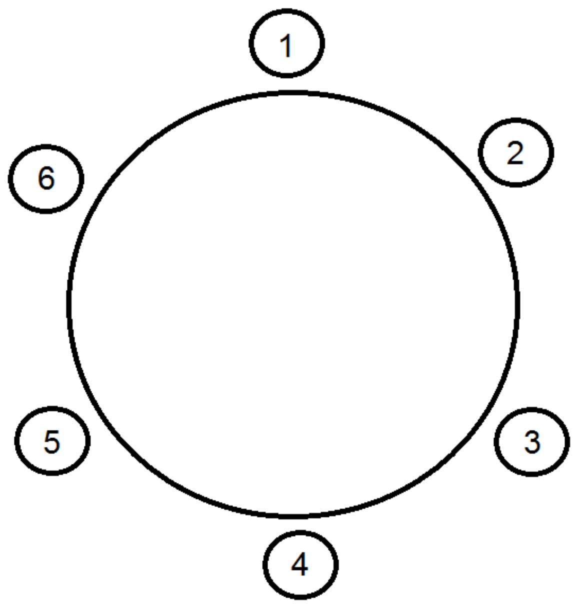
Then we started with our 3rd question. I had also discussed it in the previous session.

"Six gnomes are seated at a round table. It is known that exactly two gnomes always say the truth, and they sit side by side. In addition, exactly two gnomes always lie, and they also sit next to each other.The remaining two gnomes can both lie and tell the truth, and they do not sit next to each other. A treasure seeker walks around the table and asks the dwarves where they hid the gold. Where did the dwarves hide the gold? The answer must be justified."

- The first dwarf said that the treasure was in a cave.
- The second said - at the bottom of the lake.
- The third said - in the castle.
- The fourth said - in a fairy forest.
- The fifth said - at the bottom of the lake.

  First, I tried explaining the problem to them in this way.

"Suppose, there are 6 people seated at a round table, who actually knew about a hidden treasure. I am the detective treasure hunter, and I came to ask them about the location of the treasure."



"Now, among these 2 people, exactly 2 such people are truth tellers, that is, they will always tell you the truth, no matter what I ask them. For example, if I asked them the value of $(2 + 2)$, they will only reply 4, and nothing else. Also, it was given that these 2 people sit together.

Like this way, there are 2 other people, who just lie. They will always lie with whatever question I ask them. Also, these 2 people sit together.

And the last 2 people are of mixed type, that is they tell the truth, or lie randomly, whenever they wish to. Now, unlike the previous 2 cases, these 2 people do not sit together in the picture."

"So, now when I asked them for the location of the treasure, the first 5 people gave 5 different statements as shown above, and the 6th person stayed silent. Now the question asks that if it is possible to find the actual location of the treasure."

Jesmine then answered, "Sir, I think that that the treasure will be in the bottom of the lake. So 2 and 5 are telling the truth."

I asked, "How are you saying that? Do you have any reason?"

Arqam then asked, "So only 2 and 5 said the same statements? No one else gave the same statements?"

I replied, "Yes, you are right. Only 2 and 5 gave the same answer, but everyone else gave different answers.

Sumana then asked, "Sir, did person 6 reply anything?"

I said, "No, and that seems to be a difficult part. When I asked the same question to 6, he gave no answer."

Arqam then claimed, "2 and 5 are lying, but both of them will be of the mixed-type people."

I said, "Ok, suppose I assume that. This works because both are not seating next to each other. Now, among $1, 3, 4, 6$, two of them are just truth tellers (seating together), and the other two are just liars (seating together)."

Arqam then said, "Sir, in that case, then 1 and 6 must be the truth tellers, and 3 and 4 must be the liars."

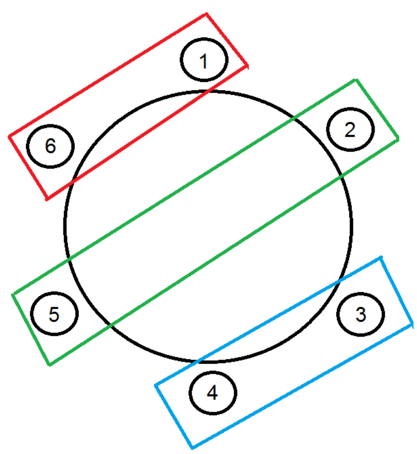
This was actually the correct answer, and Arqam found it pretty quickly by a good Trial-Error way.

Sumana then asked, "But sir, person 6 did not say anything. So how is he saying the truth?"

I said, "Yes, person 6 did not say anything. But he can be a truth teller. It is not necessary for a truth teller to answer every question, he might not answer some questions at all, but still be a truth teller."

Arqam said, "Perhaps 6 did not answer anything because he already knew the truth." This was a good argument nevertheless.

So, the final conclusion was that, $1, 6$ were the truth tellers, $3, 4$ were the liars, and $2, 5$ were the mixed-type people.



Even the seating positions seem to work out correctly. The truth tellers are seating together, the liars are seating together, and the mixed-type people are not seating together.

So, this clearly is a solution. And since from the question, only one solution should exist, we can conclude that 1 must be telling the truth, and the treasure must be lying in a cave.

But this was not complete. We still needed to show that the treasure cannot lie at the bottom of the lake, or in the castle, or in a fairy forest. Then only we will fully be able to claim that, the treasure must lie inside a cave.

In other words, we have to show that neither of $2, 3, 4, 5$ can ever tell the truth. Showing these will require showing many cases, which we couldn't fully complete in the session.

Also another important thing which I tried to say, was that, other than 2 and 5, everyone gave different answers on the location of the treasure. Now, we have to accept that the treasure cannot be in 2 different positions simultaneously.

However, the students probably did not accept it. They were arguing on whether a castle can be in a forest, so the treasure can be inside a castle, inside a forest, and like that they were making it more complicated :P.

So, then I assumed, that the treasure was in the bottom of a lake. So 2 and 5 said the truth. Now there are some more cases from here :-

(i) Both 2 and 5 cannot be the truth tellers, as they are not seating together.

(ii) Suppose both 2 and 5 are of the mixed type people. They are also not seating together, so among $1, 3, 4, 6$, two of them will be the truth tellers, and two of them will be liars. But now, all $1, 3, 4$ are telling lies, but we just have 2 liars, so this is impossible.

(iii) Suppose 2 is a truth teller, and 5 is of mixed type. Notice $1, 3, 4$ are telling lies. So for the liars to be seating together, $3, 4$ must be the liars, and 1 will be of mixed type. Also $1, 5$ are not seating together. This leaves 6, who has to be a truth teller. But $2, 6$ becomes the pair of truth tellers, and they are not seating together, so again this is wrong.

(iv) Suppose 2 is of mixed type, and 5 is a truth teller. This case was not discussed in the session, so I leave it here as an Exercise.

Exercise:- Prove that this is not possible.

Hence, from all these, we conclude that $2, 5$ cannot tell the truth, so they both must be lying.

Now, suppose just 3 is telling the truth, so the treasure is in the castle.

Then, all of the people $1, 2, 4, 5$ must be telling lies. So two of them must be liars, and two of them must be of mixed type. But this leaves 6 as a truth teller, but $3, 6$ are not seating together, so this is impossible!

The only case that now remains, is that if the treasure is in a fairy forest. We didn't discuss this case in the session too.

Exercise:- Prove that this is not possible.

There is also another case, which we didn't discuss in the class. The treasure lies in the cave, but only when 1 is a truth teller. What happens if 1 is of a mixed type person?

Exercise:- Analyse the case when person 1 is of mixed type.

So, that was all we had discussed in the session. In total we had 4 problems to be discussed, but we discussed 3 problems. The 4th problem was on an advanced game of Sudoku (**Kendoku**), and some students were not familiar with the sudoku game, so I left it. But still the whole session was fun and enjoyable.

Thank You for Reading!

PROBLEMS INVOLVING CRITICAL THINKING

Souradip Das

April 17, 2022

Session 6

About Myself: Hello! I am Souradip Das, a student of class 9. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in the 6th session.

1 Introduction

We had 2 students in the session. They were from the category of 5/6/7th graders and classmates from the same rural primary school.

We had discussed 5 problems, but unlike the first session, where I used some of my own problems, once again, in this session our Cheenta Academic Coordinator provided me with their own questions to be discussed on. So all these sessions from now on will be more like Problem Solving sessions.

2 Question 1

We started with an easy problem.

"Two kids Anya and Borya play a game. First, Anya writes a natural number on the board, and then Borya writes his number on the same board. If the sum becomes odd, then Anya will win, and if it becomes even, then Borya will win. Can one of them always win, regardless of the actions of his opponent?"

Since there were only 2 students in the session, I decided to name the players as the students only, so suppose Tuya and Rohan plays this game.

Then I started explaining the game. "So, Tuya starts the game first. So what she does is, she just writes one number on the board. For example, suppose she wrote 7."

"Then you, Rohan, will write another number on the same board."

As Rohan wanted me to write out a 9, I wrote a 9 on the board.

"So, now the game is such, that if the sum of these 2 numbers is odd, then Tuya will win. But if the sum is even, then Rohan will win."

"Now, the sum here is $(7 + 9) = 16$, so Rohan wins."

Now the question asked, ”Is there a way we can play the game, such that one of the players will always win?”

Both Tuya and Rohan replied, ”Yes”.

I asked how. After some minutes, Rohan asked, ”Do we have to write more numbers again, after writing those 2 numbers?”

I said, ”No, that is not needed. After writing those 2 numbers, we just sum them up, and if it is odd then Tuya wins, and if it is even then Rohan wins.” Then Rohan said, ”So here $(7 + 9) = 16$ is coming, so I am winning?”

I replied, ”Yes, exactly. So when Tuya wrote 7, you wrote a 9 on the board, and that is why you won. Had you wrote an 8, Tuya would have won.”

Then Tuya said, ”But if I write an even number on the board, and then Rohan writes an odd number, then I would have won.”

I said, ”Yes, but this is only for a specific case. What if you had taken a different number?”

Tuya said, ”Then shouldn’t it be the same thing?”

I said, ”Ok, suppose now you (Tuya) wrote the number 6 on the board. So Rohan, what should you write on the board now?”

Rohan replied 4.

I said, ”Correct, because now $(6 + 4) = 10$ is even, and Rohan again wins. So do you see who wins everytime?”

Both of them replied, ”Yes, Rohan is winning.”

I asked, ”Why so? Is there any reason for which Rohan is always winning?”

Rohan then said, ”So the person who writes the first number on the board, I am writing some number, which is converting their sum into an even number, so I am winning everytime.”

I replied, ”Exactly, and this is the correct approach for the problem. So, Tuya first starts by writing any number, and Rohan then writes the last number, and if the sum is even, then only Rohan is winning.” Both of them agreed with it.

”Now, as Rohan is writing next, he can already see Tuya’s number, and so he knows what number he should write next, so that the sum becomes even. And so he is winning. But, can we show that Rohan can always convert it into an even sum?”

For that, I took 2 cases. ”Suppose I take 2 cases :-

(i) Tuya writes an odd number.

(ii) Tuya writes an even number.

Now Tuya can write anything of her choice, which we cannot tell. But we can tell that any number will be either odd, or even. And hence we divided them in these 2 cases.”

”So, if Tuya writes an odd number, Rohan, what would you write next?”

Rohan said, ”I would write an even number.”

I said, ”No? then (odd + even = odd), and if the sum becomes odd, then Tuya will win.”

Rohan then immediately replied, ”Yes, then I would write an odd number.”

I said, ”Exactly, that’s it. So if Tuya writes ANY odd number, what you will do is just write another odd number, and their sum becomes even, and you will win!”

"So, if Tuya writes an even number, what will you do again?" Rohan replied, "I will then write an even number."

I said, "Yes, so it's completed. We can see that Rohan can always choose some number such that the sum is always even, and so Rohan can always win this game."

Both the students agreed to the reasoning, and the problem was fairly easy.

Then we started solving our next problem.

3 Question 2

The Question states :-
"There are 6 coins on the table: 3 heads up, 3 tails up. In one move, we can turn over any 2 coins. Is it possible to get all the coins upside down in a few moves?"

"So, there are 6 coins on a table. With 3 heads up, and 3 tails up. Now in every move, we flip any 2 coins. The question asked, if it is possible to get all the coins heads up, or tails up, in some move or not."

Rohan replied that it is possible, but Tuya said that it is not possible.

I said, "If it's possible, show me a method to do it. But if it is not possible, then explain why."

Tuya said, "So if I take 2 heads and flip them, they both will become tails, and there will be just 1 head left. Like that, if I take 2 tails and flip them, they both will become heads, and there will be 1 tail left."

I said, "Okay, but what if you take 1 head and 1 tail, and flip them?"

Tuya said, "Then. . . , here then the tail will become a head, and the head will become a tail."

It seemed like Tuya has got some idea to start this problem. A good method to start this problem is to get some ideas from the flipping experiments, and some nice observations would automatically come from there.

So I asked, "So, what exactly is happening, which is why this seems impossible to get all the coins in heads/tails?"

Tuya said, "Is it because all the coins are different?"

Then I said, "That is ok. But why is it so, that however I flip the coins, this is impossible to achieve?"

Then I gave them 3 more minutes to try out the problem.

Then, both Tuya and Rohan said, "Sir, everytime there is just 1 coin remaining, either head or tail, and it becomes impossible to flip that coin back again.

I said, "Yes, that is correct. And in fact for that reason only, this game becomes impossible. But, I can still play it in some different type of way, but still, why is it still impossible? Shouldn't there be a reason for it?"

They were still thinking on the problem, and claimed that this must have impossible for the fact, that everytime just 1 head or 1 tail remains, and it becomes impossible to flip it back.

So then I said, "Yes, and indeed these is a great observation. But if you were to explain this by writing it in a notebook, how will you write it? And how will you justify that this is impossible? We all can understand that this is impossible, but how to nicely show it?"

At this point, I gave them a hint. "So, at the start, there were 3 coins with heads up, and 3 coins with tails up. Both of these 3, are an odd number, and this is something which we might have to use."

"Now, suppose I flip any 2 coins of my choice. What can possibly happen?"

Tuya said, "The heads will become tails, and the tails will become heads."

I said, "Yes, that will obviously happen. But what exactly can happen everytime when we flip 2 coins?" We can divide them into 3 cases like this way:-
(i) We flip 2 head coins. Here, the number of head coins decreases by 2, and the number of tail coins increases by 2.
(ii) We flip 2 tail coins. Similarly, the number of head coins increases by 2, and the number of tail coins decreases by 2.
(ii) We flip 1 head coin and 1 tail coin. In this case, nothing happens! The head count and tail count both remains constant.

So, mathematically, we can show these claims in this way :-
(i) Here, we have -2H, $+2$T.
(ii) Here, we have $+2$H, -2T.
(iii) Here, nothing happens. So let it be 0.

Both the students agreed to these claims and notations of showing the head and tail counts.

Now, my purpose is to make all the coins heads up (or tails up). This can happen, only if the head count, or the tail count, becomes 0, so there should be just no heads, or just no tails.

But, in every such move we just saw, notice that the total head count can just decrease by 2, or the total tail count can just decrease by 2.

But we had started with 3 head counts and 3 tail counts, and our purpose is to make them into 6 head counts and 0 tail counts. Is this possible?

Both the students replied, "No, it is not."

I said, "Yes, as many times as we keep on adding 2 or subtracting 2 from these counts, both the counts will always remain odd. So, none of the counts can reach 0, as 0 is even.

This just shows that it is not possible to bring all the heads/ all the tails in a few moves, or this game is impossible.

Both the students agreed to this solution. They also said that the problem was not that hard, but it was nice.

So, then we moved on to the next problem.

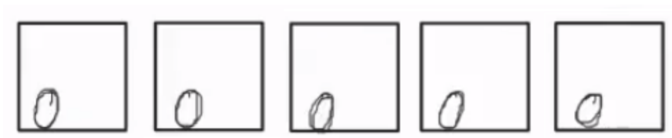
4 Question 3

Then we started with our next problem :-

"Peter has 5 cages with rabbits (the cages are in one row). It is known that each cage contains at least one rabbit. We will call two rabbits neighbors if they sit either in the same cage or in neighboring ones. It turned out that each rabbit has either 3 or 7 neighbors. How many rabbits are in the center cage?"

"So, there are 5 cages placed in a row, and each cage is filled with a specific number of rabbits, such that all the cages has at least one rabbit. (That is, no cage is empty)"

As each cage has at least one rabbit, we can put 1 rabbit in every cage for now.



I asked, "Now, there's a term "neighbour" used here. What does it mean?"

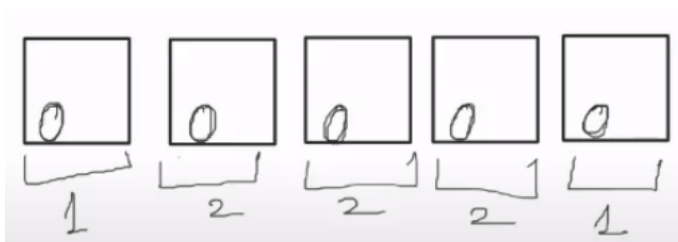
Rohan said, "It means people who live near our houses?"

I said, "Yes, but here we will define "neighbour" in terms of rabbits. A rabbit will be a neighbour of another rabbit, if either it is in the same cage of that rabbit, or it is inside a cage lying close to that cage of the rabbit."

"In other words, If I were to be a rabbit, then all the other rabbits who are lying close to me, either in the same cage, or in the closer cages, will be my neighbours."

Now, an important information was given. "Every rabbit, will either have 3 or 7 neighbours."

So, for now can keep the neighbours record for each rabbit, in each cage, in this way :-



Now Rohan asked, "But these rabbits have just neighbours in this $1, 2, 2, 2, 1$ order. They do not have either 3 or 7 neighbours."

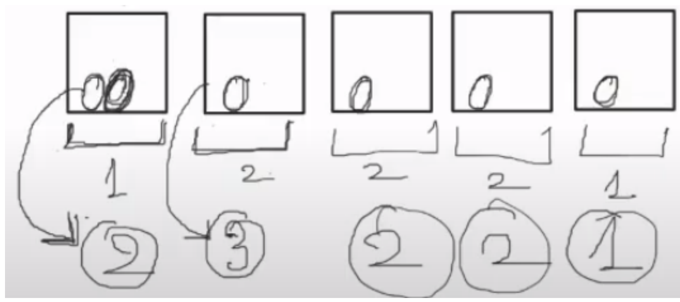
I replied, "Yes, but it may not happen that each cage will just have 1 rabbit. Every cage can have more than 1 rabbit too. But since it was given that no cage is empty, I just drew at least 1 rabbit in each cage for convenience."

At first, I gave 5 minutes time to let the students try a problem a bit. As it was a bit hard, I decided to help them.

So, I start experimenting with the number of rabbits. There was an interesting observation.

In every cage, whenever I add a rabbit, the neighbour count of that cage, as well as of the neighbour cages, will increase by 1.

For example, if I added a rabbit on the first corner cage, then the neighbour count of that corner cage, as well of the cage closer to that corner age, will increase by 1.



Also, all the rabbits in a particular cage, seems to be having the same neighbours.

So, this observation was really useful, as now we can just experiment on $1, 2, 2, 2, 1$ to get some combination of 3 or 7.

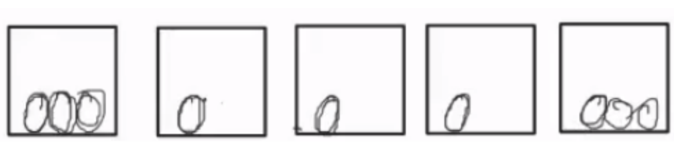
Now, suppose any of the corner cages can have rabbits with 7 neighbours. For now, if the cages are named A,B,C,D,E in that order, suppose cage A has 7 numbers. But then, cage B must have 8 neighbours, which is a problem because 8 exceeds 3 and 7, and so this becomes impossible.

So, none of the corner rabbits can have 7 neighbours, so they must have 3 neighbours each.

In other words, we found that this configuration of $1, 2, 2, 2, 1$ must be of the form of $3, a, b, c, 3$. It is also not hard to deduce that $a = b = c = 7$, but we didn't have to do it in the class.

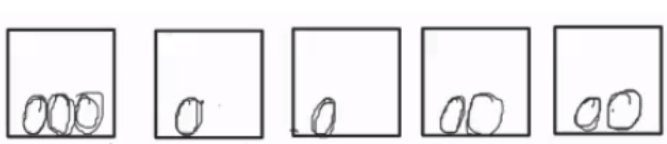
Now, there are several cases when this cornered 3's configuration occurs :-

(i) Both the cornered cages A and E have 3 rabbits each.



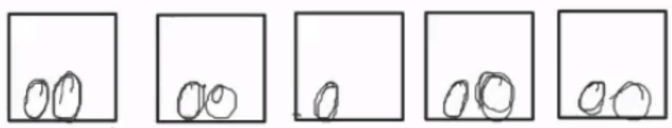
Then the neighbours count would be $3, 4, 2, 4, 3$. Now, we can only add rabbits in the middle cage C, as the neighbour count in cages A and E cannot be changed. So the best result we can get is $3, 7, 5, 7, 3$, which is not a solution.

(ii) The cornered cage A has 3 rabbits, the cage D has 2 rabbits, and cage E has 2 rabbits.



Here the neighbours count would be $3, 4, 3, 4, 3$. Again the best result we can achieve is $3, 7, 6, 7, 3$ which is not a solution.

(iii) Cages A, B, D, E has 2 rabbits each.



Here, the neighbours count seems to come up as $3, 4, 4, 4, 3$ and this was actually needed for us, because now we can just add 3 more rabbits in cage C, and the neighbours count will become $3, 7, 7, 7, 3$.

So, this clearly is a solution, and we see that the middle cage has exactly 4 rabbits. So the answer should be 4.

We kept it till here, because the problem was already becoming a bit tough for them. But the problem is not yet complete. There were still some more cases left to check.

Exercise:- Analyse all the remaining cases of the cornered 3's configuration. What do you observe?

Hint:- This is not the only solution. There are some more solutions too.

5 Question 4

This was the 4th Question :-

"The sum of five positive integers is 11. On the left side of this equality, the same numbers were covered with tablets with the same letters, and different numbers with tablets with different letters. It turned out that C + Y + M + M + A = 11. Can you tell what number is hidden behind the letter M?"

So I said, "We are adding 5 different numbers, whose sum is 11. But the numbers which are marked with the letter M, will be equal to each other. And the other numbers, which are marked as C,Y,A are not equal to each other, nor equal to the number marked M.

I asked, "Now can you find what the number M represents?"

Rohan said, "So, $5 + 6 = 11$, dosen't this work?"

I said, "No, You are adding only 2 numbers, but the question said to add 5 numbers together for a sum of 11."

So I gave them 5 minutes to try out the problem first.

After some time, Rohan said, "First, C will be 1. Then Y will be 1."

I replied, "No, that cannot happen. Then C and Y are both becoming equal, and the question had said that they all are not equal to each other."

Rohan said, "Then C is 2, Y is 1. And both the M's will be a 2."

I replied, "No, then C and M are becoming equal again.The numbers C,Y,A are not equal to any other number."

After giving some more time, Rohan replied, "Sir, I am trying the solution again. C will be 2, Y will be 3, both the M's will be a 1 each, and A will be a 4."

I said, "Yes, exactly. And this was indeed the correct answer." As :-

$$\rightarrow 2 + 3 + 1 + 1 + 4 = 11$$

Now, I showed them how to do this problem with some quick casework :-

(i) If M was 6, or more than 6, obviously the total sum exceeds 11, so this is impossible. So M should be less than 6.

(ii) If M was 5, then the sum of the other 3 numbers will be 1, which is impossible.

(iii) If M was 4, then the sum of the other 3 numbers will be 3, and it is possible only if all of them equal 1, which is not possible because all of them are distinct.

(iv) If M was 3, then the sum of the other 3 numbers will be 5.

Now, the smallest sum of 3 distinct numbers can just be $(1 + 2 + 3) = 6$, which is already more than 5. So this is also not possible.

(v) If M was 2, then the sum of the other 3 numbers will be 7. Now is 7 achievable?

The smallest sum of 3 distinct numbers can just be $(1+3+4) = 8$, because here M has already taken the value of 2, so we cannot use 2 again. And 8 already exceeds 7. So this is also not possible.

(vi) If M was 1. hen the sum of the other 3 numbers will be 9.

Here, we got the solution $2 + 3 + 4 = 9$, and so it's done. So M = 1 will be the answer.

We did not discuss a small case in the session, and that is if there exists some other solutions for M = 1 or not.

Exercise:- Are there any other such solutions that exist? If yes, then what is the total count of the no. of solutions?

6 Question 5

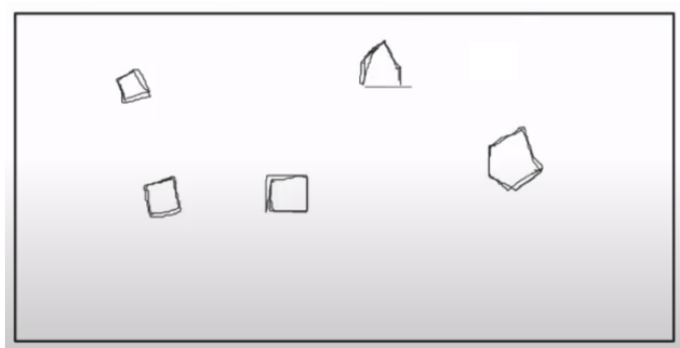
Then we proceeded with our 5th question :-
"From a big piece of paper Steve cut out 2016 shapes - squares and regular pentagons. Then Michael cut each pentagon along one of its diagonals. How many quadrilaterals were there at the end? (A regular pentagon has five equal sides and five equal angles. A diagonal of a pentagon is a segment which connects two corners that are not already connected by a side.)"

At first I explained Rohan some of the terms which were new to him. "A square is a shape having 4 equal sides and all equal angles, and similarly like that a Regular Pentagon is a shape having 5 equal sides and 5 equal angles. A Pentagon, however, is a shape having any 5 sides, which may not be equal.

Then, a diagonal of any shape would be a line joining any 2 end points, such that they break that shape into 2 parts. For example, one diagonal of a square breaks it into 2 triangles.

Then, a quadrilateral is just a shape of any 4 sides. A square is just a quadrilateral, but all the sides and angles are equal.

So I started, "So Steve had a big piece of paper, and from there he cut out small pieces of squares and regular pentagons, such that the total count of all of them was 2016. He drew the shapes in the paper in this way" :-



"Then Michael took all the regular pentagons from those shapes, then drew a diagonal, which divided those pentagons into 2 parts, and then added those parts with the squares."

"Now the question asks, how many quadrilaterals will there be at the end, after doing all these work?"

Firstly, a diagonal of a Pentagon would always divide it into 1 Triangle and 1 Square, which Rohan seemed to agree.

Then I said, "So among the 2016 shapes, there will be some number of squares, and some number of Pentagons. Then all those Pentagons are getting divided into 1 triangle, and 1 quadrilateral. After this, how many quadrilaterals would be there?"

Rohan said, "Is it 2016?"

I said, "Yes, that is the correct answer. But why so?"

Rohan said, "Because, whatever number of triangles are getting separated from the Pentagons, that number of quadrilaterals are getting counted, and that count only remains in total."

Rohan almost had the idea, and is almost correct. This is a fairly straightforward problem on the topic of Algebra, but as Rohan was not familiar with it, I decided to show this in a smaller way.

I said, "Suppose there were total 600 squares, then there would have been 1416 Pentagons. Now whenever you are breaking all the Pentagons, we would be getting 1416 triangles and 1416 quadrilaterals, so the total number of quadrilaterals becomes $(600 + 1416) = 2016$." Rohan agreed with this.

Next I said, "So if I had taken some other number of squares, like if I had taken 700 squares, then again the total number of quadrilaterals would have been 2016." Rohan again agreed with this result.

Then I asked, "Can you tell why?"

It was at this point, that I came to know that Rohan did not know algebra, so I left it here. I concluded by saying to Rohan to think about why the total count of quadrilaterals is always 2016, whatever be the count of squares I take. I also leave it here as a small exercise.

Exercise:- Prove that the total number of quadrilaterals will always be 2016, irrespective of the number of squares in the first place.

So that was all which we had discussed in the 6th Session. We had discussed all the 5 problems which we had in the problem set. Although there were just 2 students in the session, I have tried to make the session as much interactive and as much fun and enjoyable as possible.

Thank You for Reading!

SOLVING PROBLEMS FOR FUN

Souradip Das

June 13, 2022

Session 7

About Myself: Hello! I am Souradip Das, a student of class 10. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in the 7th session.

1 Introduction

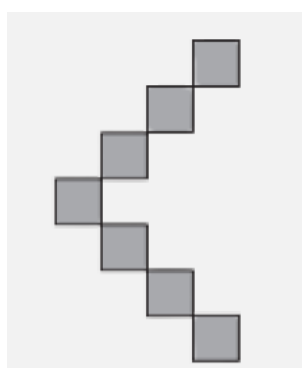
We had 4 students in the session. They were from the category of 5/6/7th graders and classmates from the same rural primary school.

We had discussed 4 problems, but unlike the first session, where I used some of my own problems, once again, in this session our Cheenta Academic Coordinator provided me with their own questions to be discussed on. So all these sessions from now on will be more like Problem Solving sessions.

2 Question 1

We started with an easy question.

"In the diagram, squares of side length 1 meet each other at their vertices. Find the Perimeter of the Figure."



At first, I asked everyone if they knew what the definition of a square is. Almost everyone had replied yes to it.

Next I asked if they knew about the meaning of the word "Perimeter", which they didn't know. So I explained :-

"If I take many different sized squares. Some squares may be big, some may be small. But the Perimeter of each square, would be just the sum of the measurement of each of its 4 sides."

"As an example, if I take a square and measured 1 one of its side length to be 5, and as all side lengths of a square are equal, the perimeter of this side would be $(5 + 5 + 5 + 5) = 20$."

Nasher asked me to explain the definition of Perimeter once again, as she joined a bit late. So I explained it to her in the same way. Hopefully the students had understood the meaning of Perimeter of a shape.

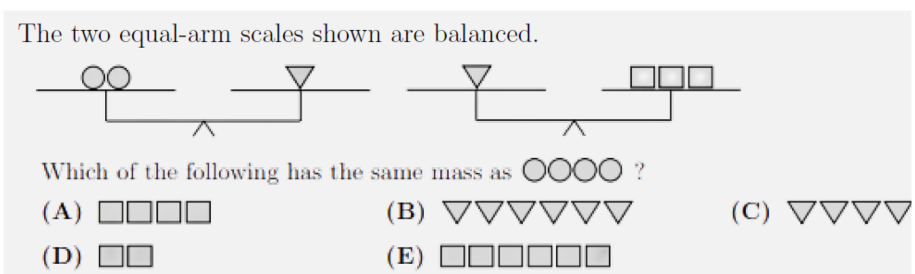
Now, back to the Problem, it was given that each of the squares has a side of length 1, and we just have to find the Perimeter of the whole figure.

It did not take for them to realize that the answer was just 28, which I didn't have to explain once again.

So, we moved on to the next problem.

3 Question 2

This was the Second Problem.



The students had already seen a weighing scale and its mechanisms, so I didn't need to explain the question much to them.

So, from the 1st balance, we have that the mass of 2 circles is equal to a triangle, or to show it intuitively, we can do :-

$$\bigcirc + \bigcirc = \triangle$$

Similarly, we can say :-

$$\square + \square + \square = \triangle$$

Everyone agreed with these results.

Next, I argued in this way. "Since we already have that a triangle has mass, same as of 2 circles, as well as of 3 squares, what can we get?"

Nasher said, "So we can say that 2 circles would equal 3 squares?"

"Exactly." I said. So we can say :-

$$\bigcirc + \bigcirc = \square + \square + \square$$

There were no doubts till this point. So then I came back to the question, "So which of the following options given in the question (either made of squares, or triangles) would equal the mass of 4 circles?"

At first, everyone of them answered B. So they all meant that 4 circles would equal 6 triangles.

Nasher then once said it could be A, but then changed her option to B.

Before I started to talk, Rohan quickly changed his option to D.

Masud once said the answer would be E.

Now I started to argue, "I knew that :-

$$\bigcirc + \bigcirc = \triangle$$

So, that means :-

$$\bigcirc + \bigcirc + \bigcirc + \bigcirc = \triangle + \triangle$$

Or, Mass of 4 triangles will equal that of 2 circles, but option B says 6 triangles, so this is wrong."

"Next, the mass of 2 circles already equals the mass of 3 squares, so the mass of 2 squares is very less than that of 4 circles. D is also out of the question."

After this, Rohan replied that the answer would be E.

So, I came to option E. "For Option E, it says that 4 circles would equal 6 squares." I asked, "Is this true?"

Both Rohan and Nasher said, "Yes it is, because 2 circles equal 3 squares, so 4 circles will equal 6 squares."

So, the answer would be (E). According to everyone's opinions, this wasn't a very hard problem.

4 Question 3

The next problem was long, and slightly confusing.

"A group of friends are sharing a bag of candy.

(i) On the first day, they eat $\frac{1}{2}$ of the candies in the bag.

(ii) On the second day, they eat $\frac{2}{3}$ of the remaining candies.

(iii) On the third day, they eat $\frac{3}{4}$ of the remaining candies.

(iv) On the fourth day, they eat $\frac{4}{5}$ of the remaining candies.

(v) On the fifth day, they eat $\frac{5}{6}$ of the remaining candies.

At the end of the fifth day, there is 1 candy remaining in the bag."

How many candies were in the bag before the first day?

I was also aware, that the students were not familiar with algebra, so I was supposed to do this problem without specifying any variables.

First, to make a sense of what the problem was stating, I tried explaining it this way,

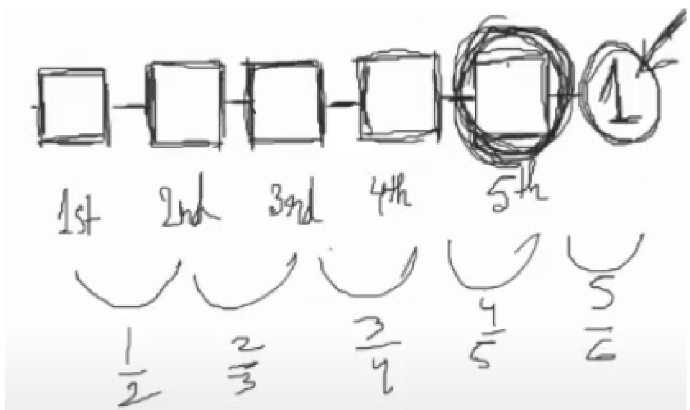
"So, instead of "a group of friends", just assume there's a single friend. Suppose I am the only person who has candies, and I don't know the count, I have lots and lots of candies."

"Now, in the first day, what I do is, eat $\frac{1}{2}$ of the total number of candies I had. So now I have $\frac{1}{2}$ of the candies left. In the next day, from this half no. of candies which was remaining, I ate $\frac{2}{3}$ of those total number of candies, and this pattern continues and so on. At the end of the 5th day, only 1 candy was there inside my bag." So the question simply asked to find the number of candies I originally started with.

Several students asked me once to explain this again, so I tried explaining it again, this time with a circle (denoting the no. of candies), and then halving it and eating one half in the first day, and then dividing the other half 3 times, and so on. They hopefully understood the problem from here.

So, I let them try this problem themselves for 5 to 7 minutes. After some time, only Rohan answered that the answer would be 1440.

As a lot of time passed, I drew out a sketch of the no. of expected candies in each day, when each operation was simultaneously taking place.



The boxes where shown as the particular day, and the fractions below showed the respective operations, and we want just 1 candy at the end of 5th day.

Now to start with, I asked, "What should I put in the 5th box? That is, how many candies should I have, such that if I divided the count into 6 parts, and ate 5 parts of it, then only 1 candy will remain?"

Rohan first asked if 1 candy is enough. Everyone else immediately realised that you cannot really divide 1 candy by 6, it might be divided into smaller candies, but not the total count.

Next Rohan asked if 3 candies were enough. Here the students at first agreed to this, as they thought that they can divide 3 candies into 6 parts, by breaking each candy into 2 halves. But then I explained the misconception that the candies themselves cannot be divided into. They can be, but according to the question, it is the total count of the candies we are dividing, not the candies themselves.

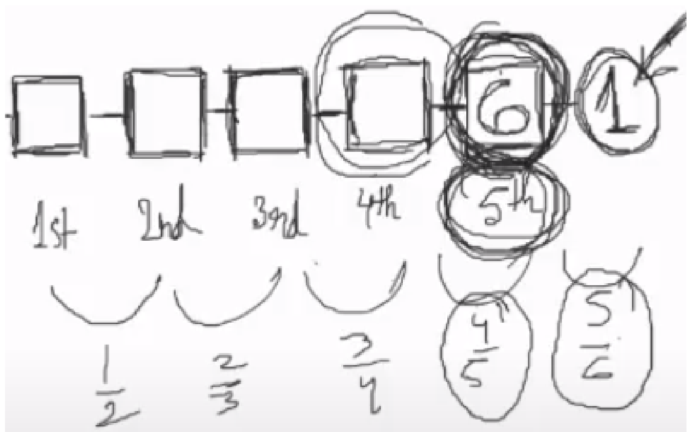
Note:- In general, it is totally allowed to divide the candies itself (here comes the concept of fractions), but here it will still not work.

Exercise:- Show a rigorous explanation of why 3 candies actually is not the answer.

Next, Nasher immediately came up with the idea that the if she just took 6 candies, divided them in 6 parts, so each part contains 1 candy, and now if I eat 5 of those parts, then only 1 will remain.

And this was exactly the answer. So there will be 6 candies in the 5th box, or in other words, at the 5th day, I had exactly 6 candies.

Now we go to the 4th box. The question would be almost the same, but the only the values change a bit. I asked, "So, how many candies should I have at the 4th day, such that if I divide their count into 4 equal parts, and eat 3 of those parts, then only 6 candies will remain? (when we move to day 5)"



Nasher at first answered 10, arguing in the way that eating up 4 candies from these 10, would leave 6 candies only. Here I told her that she was at first dividing these 10 candies into 5 equal parts, then eating up 4 of those 5 parts. So in this case, each part would have 2 candies, and we are eating up 4 parts, so $(4 * 2) = 8$ candies was consumed, so only 2 candies remained.

The misconception was that she took the she did not divide the candies into 5 equal parts, at the first place, which was actually necessary.

After some time, Nasher answered 16, but then quickly realised that 16 cannot be divided into 5 parts. (Although it is actually possible).

Next, Rohan answered 20, but then I showed that for 20 candies, each part would have 4 candies, so if I ate 4 of those parts, I had basically eaten $(4*4) = 16$ candies, so only 4 candies would have remained. This is also not the answer as we wanted 6 candies instead.

Masud then answered 30, and this was indeed the correct answer.

Dividing 30 candies into 5 equal parts, meant that each part would have 6 candies, and we ate 4 of those parts, so we ate up $(6 * 4) = 24$ candies, so 6 candies would remain. So we would want to start with 30 candies at the 4th day.

By now, a pattern had already started emerging, and as Rohan correctly spotted out, he immediately answered 120 candies for the 3rd day. This was indeed correct, but to get to the answer, especially the pattern which came out was :-

We can split out 120 into 4 parts as $\{30, 30, 30, 30\}$. We took off 3 of those parts, so 30 would remain, simple and straightforward!

Similarly, we would want 360 candies in the 2nd day, the same idea, by breaking it up into 3 parts as $\{120, 120, 120\}$, then we remove 2 of those parts, so only 120 remains.

So, finally we would have 720 candies in the 1st day, and the whole sketch comes down to this :-

$$720 \rightarrow 360 \rightarrow 120 \rightarrow 30 \rightarrow 6 \rightarrow 1$$

Which indeed works if we intend to double check with all the operations.

So, the final answer, or the total no. of candies we initially started with, is 720.

Now all these discussions was just a logical way of getting the answer, without using the concept of Algebra. So for a small Exercise :-

Exercise:- Solve the same problem rigorously, using the concepts of Algebra.

Note:- A surprising fact in my opinion about this problem is that, the algebraic method would be a lot more tedious. What I had discussed in the session itself was actually a technique of "Working Backwards", where the end result is given, and we keep on proceeding backwards to get the initial answer. This indeed, is actually is the most effective method to do this problem.

5 Question 4

This problem that we discussed, was based on observations.

A robot is placed on the grid shown. The robot starts on square 25 , initially facing square 32 . The robot (i) moves 2 squares forward in the direction that it is facing, (ii) rotates clockwise 90°, and (iii) moves 1 square forward in the new direction. Thus, the robot moves to square 39 , then turns to face square 38 , then moves to square 38 . The robot repeats the sequence of moves (i), (ii), (iii) two more times. Given that the robot never leaves the grid, on which square does it finish?

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

So, instead of just considering a robot (which they may not know), suppose I, myself, is standing on square 25.

At first, I was initially facing box 32, and as 32 is just below 25, this statement just means that I was facing south.

So, what then I did was I moved 2 steps forward, so as most of the students figured out, I would then land on box 39.

Next, I moved to box 38. Now this means, that I rotated myself 90° clockwise to face box 38. At first, I asked everyone if they knew about the meaning of the word "clockwise".

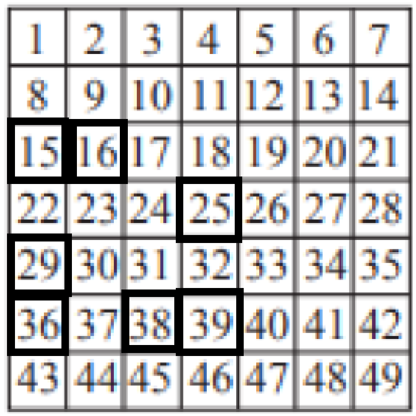
The definition of clockwise was just moving in the direction like the hands of a clock. Nasher knew about it, and otherwise, for an overall explanation, I showed it by drawing out a small clock. And Similarly, anti-clockwise would mean its exactly opposite direction.

So, now once I turned to square 38, I directly went to square 38. So that counts as a 1 step forward.

So this was the total sequence of moves I did :-

Initial Direction → Move 2 steps → Rotate 90° Clockwise → Move 1 step

Now this sequence, as stated, is to be repeated 2 more times. At first the students were taking some time to get the situation, but once they got the idea, everything was straightforward. This will be the sequence of moves in the square grid :-



So, the last ending square, will be 16.

This problem, unlike the other problems, required no knowledge of techniques nor any calculation, but just observations from the square grid, and getting the right idea of the operations which were to be done in the grid, which can be figured out by just logic.

So that was all which we had discussed in the 7th Session. We had discussed all the 4 problems which we had in the problem set. There were 4 students in the session, and I have tried to make the session as much interactive and as much fun and enjoyable as possible.

Thank You For Reading!

PROBLEMS INVOLVING DEEP THINKING

Souradip Das

June 13, 2022

Session 8

About Myself: Hello! I am Souradip Das, a student of class 10. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in the 7th session.

1 Introduction

We had 1 student in the session, Rohan. He was from the category of 5/6/7th graders from a rural primary school.

We had discussed 3 problems, but unlike the first session, where I used some of my own problems, once again, in this session our Cheenta Academic Coordinator provided me with their own questions to be discussed on. So all these sessions from now on will be more like Problem Solving sessions.

2 Question 1

We started with a very nice Conceptual Question :-

"Arrange 5 points A,B,C,D and E on a plane so that you can point exactly 8 triangles with vertices at the marked points. List these triangles."

At first, I asked Rohan if he knew about the definition of points. He was not aware of the meaning, so I told him that points were nothing but dots (as I showed them on the board). A more clear definition of a point, would have been that, it is a 0 dimensional object just specifying a position in a plane, but we didn't need to discuss it.

And then from there, I brought up the definition of a line (specifically a straight line) as a stack of dots, hence points. A lot of points joined together, in a row, would give me a line. This becomes a 1 Dimensional object.

Then, lots of lines together, would give me a 2D plane. We didn't go to the idea of a plane yet, but we discussed that a number of lines, if joined together in some way, would give me a shape, like a square.

After this, I asked Rohan what would emerge if a lot of squares are joined together. He said that a rectangle would form and he was also right, (as joining squares side by side through the same plane, would indeed form another 2D shape, and here it turns out as a rectangle) but I told him that we would join them in some different way.

"If, suppose I join squares in this forward and sideways motion, I get some different shape."

Rohan said, "It turns out like a book or a box."

I said, "Exactly, and this is just a box only. But as a name, we call it a cube. Have you heard of a cube?"

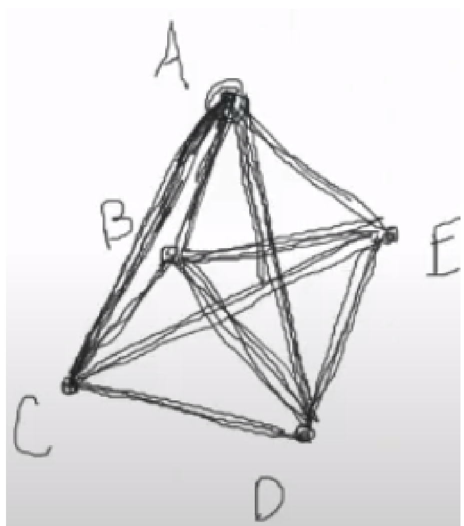
Rohan said that he had heard of the Rubik's Cube. Indeed, it's a cube, but the standard $(3 * 3 * 3)$ cube, is made up of even smaller cubes. I showed him the pictures by going through google.

So, on putting up lots of squares, we get a cube, or a 3D shape.

Or, to be more precise, putting up lots of 2D planes form the 3D plane.

After this discussion, we moved back to the problem. So, we are given 5 points/dots, somewhere (in just one 2D plane), and now we have to arrange them in some way, such that if I join all the pairwise points, exactly 8 triangles would form.

As an example, I took this configuration of points :-



After some systematic counting, we found out that 10 triangles were formed, which is more than 8 triangles.

Exercise: Is there an easier/systematic way to get to the count of triangles?

So, this arrangement does not work. Then how could we possibly decrease the count of the triangles?

I quickly took another arrangement of 5 points, and again it turned out to have 10 triangles only. It seemed at first that the no. of triangles will always be 10.

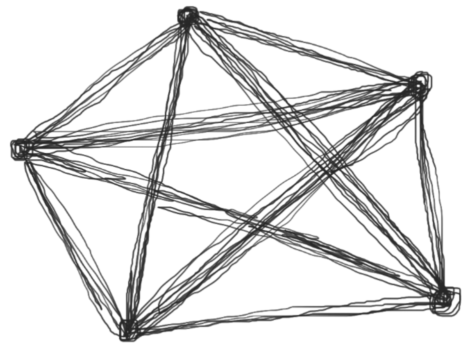


Figure 1: Again, 10 Triangles were formed

So, is 8 triangles never possible? Or is it possible to decrease the count of triangles in some way?

I started with a fact, and then with a question.

If I try arranging 2 points, joining them would always give a line, unless the 2 points coincide, which isn't allowed.

So, for the case of 2 points, only a line is formed. For, the case of 3 points, a triangle should form. But is it possible to get an arrangement of 3 points, such that no triangle forms?

Rohan correctly spotted it, and answered that no triangle would form, if the points are placed in a row. Instead, in this case, only a straight line is formed. These points have a special name, and we call them collinear points.

So we got some idea of a technique here. In the earlier 2 figures, we did not take the case of any collinear points, so perhaps that is why we were already having 10 triangles as our answer. But now, can we use this idea to decrease the count of triangles? And if yes, then how?

Rohan immediately suggested to take a figure where the points A, B, C are in a line, and points D, E are placed elsewhere. And this is the perfect way to start using this idea.

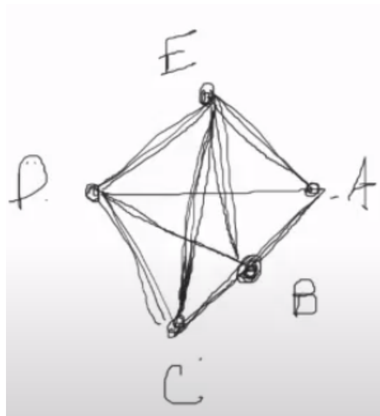


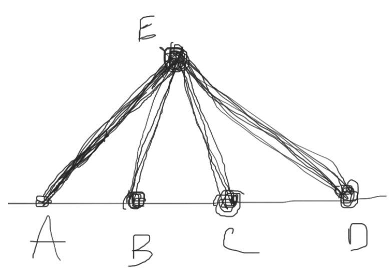
Figure 2: Here 9 Triangles were formed. We are so close!

Once again, after some counting, we got a total of 9 triangles. So close! but still not the answer we were looking for.

Exercise:- Is there an easier/systematic way (like the same Exercise above) to get to the count of triangles?

So, is this a dead end?

Rohan disagreed, and continued to try the problem. There would still be the case of taking 4 points A, B, C, D on a single line (that is, 4 collinear points) and the 5th point E located elsewhere, and he was absolutely right. So we checked that case :-



Here, after some counting, we could count only 6 triangles. And this was actually expected, as making a set of points collinear, is drastically reducing the number of triangles, and it got reduced from 9, to just 6 triangles.

Exercise:- Is there an easier/systematic way (like the same Exercise above) to get to the count of triangles?

And, obviously making 5 points collinear would create 0 triangles. So we need not check that case.

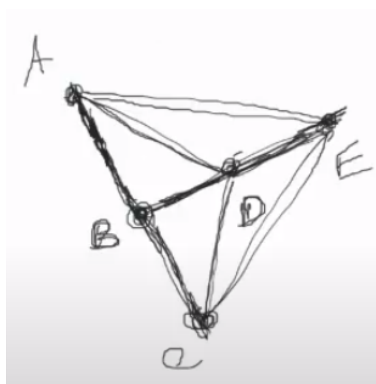
So, after analyzing all the cases, we made a table of the results that we got till now :-

(i) No points are collinear. Then there are 10 triangles.
(ii) If 3 points are collinear, then there are 9 triangles.
(iii) If 4 points are collinear, then there are 6 triangles.
(iv) If 5 points are collinear, then there are 0 triangles.

Now the question comes. Are all the cases checked? Is this the dead end? Is it impossible to bring 8 triangles instead?

Rohan at first said that it was impossible. But once I started drawing a figure, he immediately answered out.

We had missed 1 case! There can be a case, where 3 of the points A, B, C are collinear, and another set of points B, D, E are collinear. This is totally possible, and we did not see this case yet, as it is involving 2 **different** sets of collinear points.



Next, we started counting the number of triangles. And as this was the final part of the suspense, I will exclusively show the triangles that we counted. So we counted :-

$\triangle$ABD $\triangle$CBD $\triangle$ABE $\triangle$CBE $\triangle$ADE $\triangle$CDE $\triangle$ACD $\triangle$ACE

For a total of 8 triangles.

There it is, **THIS IS THE BREAKTHROUGH**! This was what we were actually looking for, and we have finally found the configuration, involving 8 triangles.

So, we can finally include our 5th case :-

(v) If we have 2 lines, each containing 2 different sets of 3 collinear points, then there are 8 triangles.

Another important thing to note here, is that there are no other cases left! We cannot simply have 2 lines, each containing 4 collinear points, or 1 line containing 4 collinear points, and another containing 3 collinear points. This is simply not possible.

Hence, the final answer, would be to consider 2 lines of 3 collinear points each, and that gives us a total count of 8 triangles.

I really liked this problem, although I had actually solved it before the session. What I like about this problem, in my opinion, is that everyone has to first think of an idea, a non trivial idea, and analyze every case or possible configurations that come in the way. This is a perfect example of a research based problem, simple yet so conceptual, involving deep thinking, and continuously trying out every possible idea which might have struck. And indeed, it is the very breakthrough in the end, just like the accomplishment of a goal, which makes it enjoyable.

3 Question 2

Next, we moved on to another nice problem.

"During the first half of the year, the lazy Pasha forced himself to solve problems in mathematics. Every day he solved no more than 10 problems, and if on any day he solved more than 7 problems, then for the next two days he solved no more than 5 problems a day. What is the largest number of tasks could Pasha decide in 7 consecutive days?"

So there is a boy named Pasha who used to solve math problems everyday. As a limit, everyday he would solve at most 10 problems. But as lazy as he was, if on some day, he solves more than 7 problems, then on the next 2 consecutive days, he won't solve more than 5 problems a day. Given these conditions, we have to find the maximum number of problems he could possibly solve, in 7 consecutive days, or basically a week.

At first, Rohan did not understand the condition of Pasha's laziness. So I explained it a bit :-

Suppose the days are in this order, Sun. Mon. Tues. up to Saturday. Suppose at some day, say Monday, he did 8 problems. Since he did more than 8 problems on Monday, by that condition, it would mean that on Tuesday and Wednesday, he will do at most 5 problems.

After this, Rohan hopefully could understand the problem. Then he gave me an example of a possible sequence of problems done in each day of a week :-

$$7 \to 8 \to 5 \to 5 \to 7 \to 7 \to 10$$

Then, he also argued that the 8 in the second place can be replaced with 10, and nothing would change, and this was indeed the right thing to do (which he also realized himself), that we want to maximize the number of problems done. So we get this :-

$$7 \to 10 \to 5 \to 5 \to 7 \to 7 \to 10$$

Where, the total number of problems result to 51. So if Pasha does the problems in this order, then he would be able to solve 51 problems in a week.

But, is this the maximum value? On asking this to Rohan, he thought for a minute and said that 51 would be the answer, but he hesitated and told me to wait.

After waiting for some time, he claimed that he got a solution larger than 51. This was his solution :-

$$7 \to 7 \to 7 \to 7 \to 7 \to 7 \to 10$$

Giving us a total sum of 52. And then he also claimed simultaneously, that 52 would be the answer.

But here, I argued that there can be any answer more than 52. I claimed, "Maybe 53 problems can also be solved, maybe 54, how are you sure that 52 is the maximum value?"

For a start, I said, suppose the sum could be more than 52. For that, at least one of the 7's must be increased by its value.

Then Rohan answered that if we are increasing one of the 7's then by the conditions, on the next 2 days Pasha would be able to do at most 5 problems.

And this was exactly the correct idea. If, I increase one of the 7's, maximum to a 10, then for the next 2 days Pasha cannot solve more than 5 problems. This means we are basically changing A row of 7's :-

$$7 \to 7 \to 7$$

To :-

$$10 \to 5 \to 5$$

But, $(7+7+7) = 21 > (10+5+5) = 20$, so we are actually decreasing the total count by 1. So, increasing the value of a 7 is not working at all.

And since we have no other option to increase the count, we can confirm that the maximum value we can achieve is 52.

Exercise:- There are 2 small cases (on changing of 7's) which I missed in the session, which also needs to be checked once before this problem becomes complete. What are those cases?

4 Question 3

Next, we moved to a much harder problem.

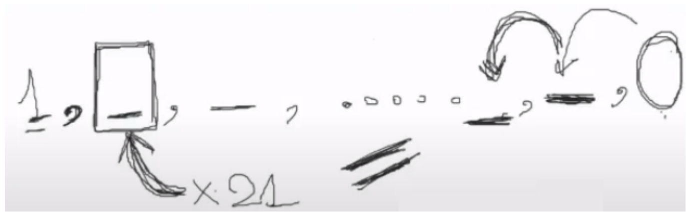
"Masha wrote down on the board in ascending order all the natural divisors of a certain number N (the very first divisor written out is 1, the largest divisor written out is itself the number N). It turned out that the 3rd divisor from the end is 21 times larger than the second one from the beginning. What is the largest value can N take?"

At first, I had asked Rohan if he knew about factors of a number. As he knew about them, I decided to discuss this question a bit.

So, Masha is a girl who took some unknown number N which was not known to us. What she did was, she listed out all the factors of N one by one, and she noticed that the 2nd factor, if multiplied by 21, was exactly equal to the 3rd last factor of that number.

Rohan was taking some time in understanding the question, so I gave him an example :-

Take 12. If we list out the factors of 12, we get :-



$$1, \underline{2}, 3, \underline{4}, 6, 12$$

So, 2 becomes the 2nd factor, and 4 becomes the 3rd last factor. But here, we get that $\underline{2} * 2 = \underline{4}$ So, 4 (or the 3rd largest factor) is 2 times that of 2 (the 2nd factor).

Masha did the exact same process for her number, but what she found out is that 3rd largest factor would be 21 times that of the 2nd factor. In the previous example it was 2 times.

At first Rohan told me to take the 2nd factor as 3, so automatically the 3rd last factor should be 63.

But then, he changed his mind and put 2 as the 2nd factor instead, so the 3rd last factor becomes 42.

The first number to check this out was 42. The factors of 42 are :-

$$1, \underline{2}, 3, 6, 7, 14, 21, \underline{42}$$

Here, 42 occurs as the last factor, but we want it as the 3rd last factor.

Rohan then told me to check 84, and this was very logical as 84 is the second number after 42, divisible by 42. The factors of 84 are :-

$$1, \underline{2}, 3, 4, 6, 7, 12, 14, 21, 28, \underline{42}, 84$$

Here, it seems like 42 is the 2nd last factor.

We next checked 126, the 3rd multiple of 42. It's factors are :-

$$1, \underline{2}, 3, 4, 6, 7, 9, 14, 18, 21, \underline{42}, 63, 126$$

Here, we finally got a solution. The number 126 satisfies the conditions.

But then, is this the largest? I argued that this indeed is a solution, but is it the largest possible solution? Can there be a larger number possible?

The problem was, we got only 1 solution.

Then, I tried to bring the idea that the 2 may not be the 2nd starting factor, it can be 3, or 5 as well.

Rohan then said that it can be 10 also. But the 2nd factor cannot be 10, as the primes 2 and 5 dividing 10, would also sit before 10.

$$1, 2, 5, 10, \cdots$$

At this point the discussion wasn't going very interesting, and since I was almost out of time, I had to close the session.

I left 2 questions to Rohan for this problem :-

(i) Can 5 be the 2nd factor of this number?

(ii) We got that 126 is a solution. But this was not the answer we were looking for. Is this the highest solution? If not, what is it?

(**Exercise**:- What do you think?)

I highly recommend every reader (assuming that they know the basics about factors of integers) to actually try this last problem. It's a wonderful problem, as much hard as the 1st one, in fact much harder than that. Otherwise, it is very non-trivial, the answer is not at all obvious, plus it cannot be solved by just normal methods. And, a large part of the main idea was not even discussed in the session. I just had a friendly discussion with Rohan, upto to the part which was understandable for him.

Exercise:-

(i) Are there infinitely many solutions for N?

(ii) If No, can you find all the possible solutions of N? What is the largest N?

(iii) Can you prove all of these claims rigorously?

So that was all which we had discussed in the 8th Session. We had discussed 3 of the 6 problems which we had in the problem set, and the last problem we did, was not fully discussed. And, I have tried to make this session as interesting, informative, and as enjoyable as possible.

Thank You For Reading!!

SQUARES, CIRCLES AND FACTORS

Souradip Das

June 14, 2022

Session 9

About Myself: Hello! I am Souradip Das, a student of class 10. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in the 9th session.

1 Introduction

We had 2 students in the session, Rohan and Gour. They were from the category of 5/6/7th graders from a rural primary school.

We had discussed 3 problems, but unlike the first session, where I used some of my own problems, once again, in this session our Cheenta Academic Coordinator provided me with their own questions to be discussed on. So all these sessions from now on will be more like Problem Solving sessions.

2 Question 1

We started with a good question.

"The diagram consists of a 4-by-4 square divided into 16 unit squares, and all the diagonals of these 16 unit squares. How many squares of all sizes and positions are there in this diagram, including squares that are made up of other squares?"

Firstly, the question was a bit confusing as a diagram was mentioned in it, but no diagram was actually given to us. So we constructed a diagram ourselves.

Next, I asked Rohan if he could understand the question. Rohan could recall (from his previous sessions) that a square is a 4 sided polygon, with equal sides and corners (which actually meant that each angle is a right angle).

Now, I said, "A 4-by-4 square, is nothing, but a square, which, if I start dividing it with line segments in this way, would form 16 equal smaller squares, in this way."

Rohan replied that there are $4 * 4 = 16$ squares. I agreed, and said that it is due to my drawing that the shape of the squares are looking a bit different, but actually all the 16 squares are of same shape and of equal length.

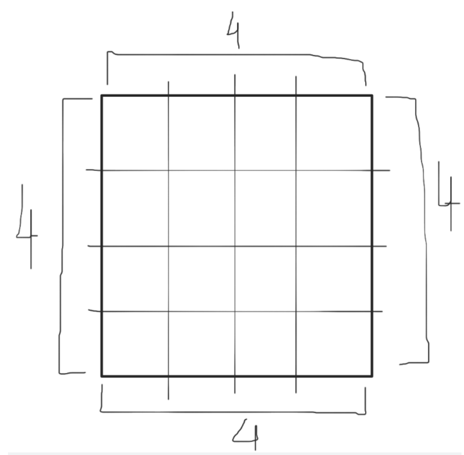


Figure 1: This is how a 4-by-4 square would look like.

Next, I asked Rohan what a diagonal is.

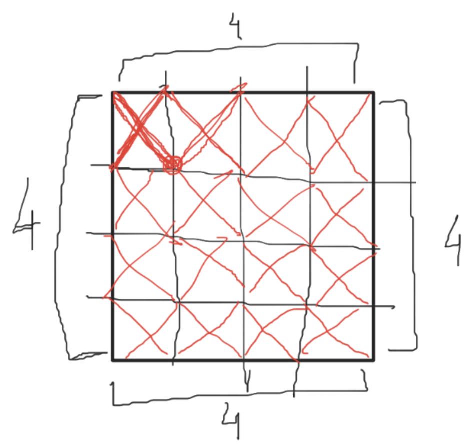
Rohan thought for some time and asked if it is a shape having 5 sides.

I corrected him by saying that what he was thinking of, is actually a Pentagon. So a shape, or specifically, a polygon, having 5 sides (of any length) will be a pentagon.

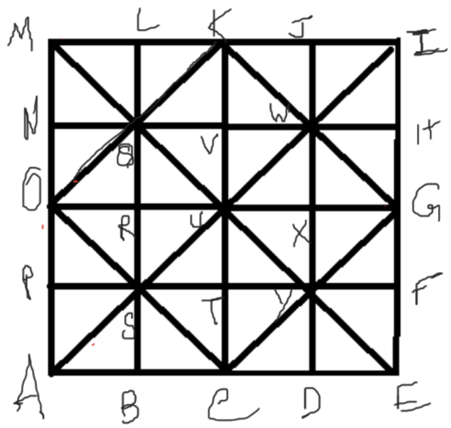
And, a diagonal is simply a line segment joining 2 opposite, to be specific, 2 non-adjacent points of a polygon. It is just like a line from 1 point to another, such that those 2 points are far away, or, do not share any side with each other.

We already had a discussion about these with Rohan in some of the previous sessions, so hopefully he could recall these topics from here.

Moving on, it was mentioned that all the diagonals of these 16 small squares are drawn. So the final picture would be that of crosses over all the squares, somewhat in this way.



Once again, it was due to my freehand drawing that the squares, and the diagonals, were not looking accurate. So I made a new diagram and we started discussing on that.



The only small mistake from my part, which went unnoticed in the session, was that I forgot to draw all the diagonals. So we actually solved a different question, and hence had a different answer.

So, we started counting the squares.

Rohan started by considering the smallest squares ($1 * 1$ type), for a total of 16 in number. Then Rohan counted a total of 7 medium squares ($2 * 2$ type), but he was missing 2 of the squares, so overall there were 9 medium squares. Next, there were 4 large squares ($3 * 3$ type), and of course the original square ($4 * 4$ type) from which we started with, is also counted.

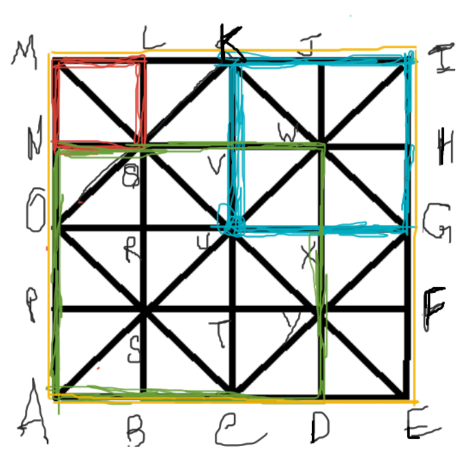


Figure 2: Different Sized squares, shown by different colours.

That makes a total of $(16 + 9 + 4 + 1) = 30$ squares.

Next, as Rohan pointed out, we also missed out some of the squares. Those are drawn diagonally, in an angle.

There are 4 such medium squares, and 1 such large square.

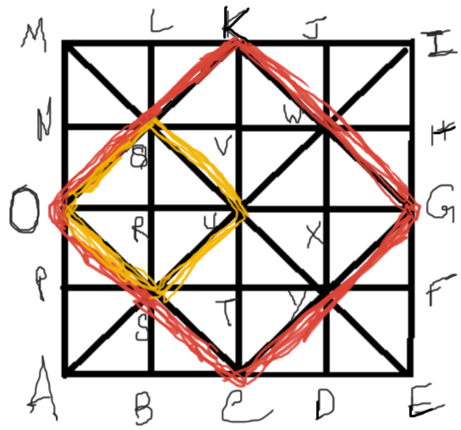


Figure 3: Squares drawn at an angle, shown by different colours.

That gives a total of 5 more squares. So we got an answer of 35 squares.

Since I had misinterpreted the question (during the session), I was actually confused on why the intended answer was not coming. So I kept it out this way, and claimed the answer to be 35. You, the readers, can try the actual problem as an exercise.

Exercise:- Can you solve the original problem?

3 Question 2

The second question that we discussed, was much tougher.

Eight semicircles line the inside of a square with side length 2 as shown. What is the radius of the circle tangent to all of these semicircles?

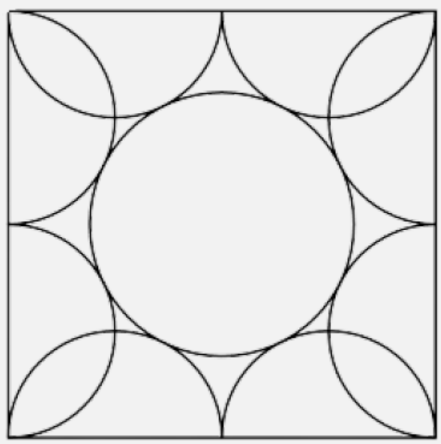
(A) $\frac{1+\sqrt{2}}{4}$

(B) $\frac{\sqrt{5}-1}{2}$

(C) $\frac{\sqrt{3}+1}{4}$

(D) $\frac{2\sqrt{3}}{5}$

(E) $\frac{\sqrt{5}}{3}$



At first I asked Rohan if he knew about a semi-circle. He was not aware of it. But he knew a circle, which is just a round shape as already shown in the middle of the figure.

And, dividing a circle by a straight line (specifically, a diameter), into 2 equal halves, will give us 2 semi circles. So a semi circle is just a half portion of a circle. Hopefully Rohan was able to understand a semi-circle, intuitively, once I showed him a figure.

And, a radius would simply be the line segment joining the center through the side of the circle. Rohan claimed that it would look like the tire of a bicycle, and he was exactly right. And moreover, if a circle has a radius, then so does a semi-circle. And, all the radii of a particular circle (or a semi-circle) will be of the same length, because it remains constant.

Next, we went back to the problem.

We have a square having 4 equal sides of length 2. And, 8 equal semi-circles are drawn as shown, on the side of the square. Finally, a circle is drawn in the middle, just touching all the semi-circles. The question asks to find the radius of the circle.

Rohan started with a very nice claim. Since exactly 2 equal semi-circles are placed on each side of a square of length 2, the length (or, specifically, the diameter) of each of the semi-circle, should be 1.

Again, Rohan, after some tries, himself finds out that since this diameter has a length of 1, the length of the radius (that is, the distance from the center to the semi-circle) of the semi-circle, would be just the half of 1, which is just $\frac{1}{2}$.

So, he figured out that the radius of each semi-circle, would be $\frac{1}{2}$.

After drawing out the centers of each of the semi-circles, as well as the circle, we roughly get this configuration :-

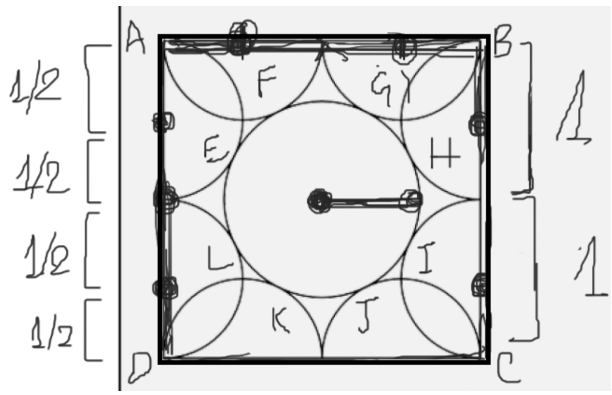
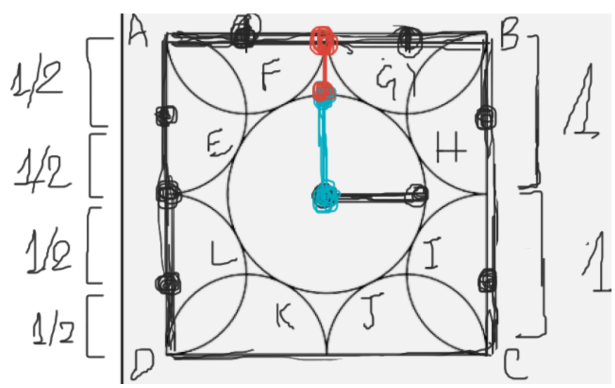


Figure 4: Radius of each of the semi-circles is $\frac{1}{2}$.

This was a great start by Rohan. Now, I asked, "How do we proceed next? We found out the radius of each of the semi circles, which is $\frac{1}{2}$, but we are supposed to find the radius of the middle circle situated there."

At first Rohan claimed that the radius of the middle circle is also $\frac{1}{2}$, particularly because it looked like so in the picture, plus he tried justifying it in a different way.



He claimed that the line which, if I draw from the center of the circle to the middle point of AB, can be divided into a red part, and a blue part.

He said, as the blue part is the radius of the circle, it should be $\frac{1}{2}$.

I agreed with all the claims that he made above, but then I asked how exactly he could claim that the blue region would be of length $\frac{1}{2}$. Also it looks almost as $\frac{1}{2}$ from the picture, but it is still not convincing, and it has to be shown mathematically.

So I asked, "How are you saying that it will be exactly $\frac{1}{2}$? It might happen that the red part is a bit lesser than $\frac{1}{2}$, and the blue part will be a bit more than $\frac{1}{2}$, or vice versa. But if the blue part was exactly $\frac{1}{2}$, then so would be the red part, as the sum of them would be 1."

Then I said Rohan that this problem cannot be solved so easily, it needs an extra topic to be covered.

On asking Rohan, he knew about the definitions of acute angles, right angles, obtuse angles and so on. So it was not hard for me to explain the definition of a right triangle, which is just a triangle with a right angle.

Next, I was supposed to explain Pythagoras Theorem to him, as it was necessary to solve this problem. On asking what it is, I said that it's a theorem, which states an important result on right triangles, but I didn't state it, as he might not be able to understand it at this point.

Plus, I was also aware (from some of the previous sessions) that Rohan was not that familiar with algebra, or with the use of variables, which is required to continue with the discussion of the Pythagoras Theorem, and also required to solve the current problem. So I kept the discussion till here, and moved on to the next problem.

As for the readers, I can keep this problem as an Exercise.

Exercise:- Can you continue further and finish this problem?

4 Question 3

The next question that we discussed, was nice (although well-known) and conceptual.

"A four digit number is said to be cool if it is completely divisible by the numbers from 1 to 10. How many such cool numbers are there?"

Also, Gour had joined from this point of the session, probably because she forgot that we had a discussion that day. That is ok (as recordings are usually available), so we started with our problem.

Firstly, a 4-digit number, is just a number, having 4 digits. For an example, I took the number 1825, which has 4 digits.

Now, a 4-digit number is called to be "cool", if I can divide this number with each of the digits $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then there should be no remainder.

For example, suppose I take the same number 1825. Now, 1 divides 1825 (in fact, 1 divides all possible numbers), but 2 does not divide 1825. So 1825 is not actually divisible by all the numbers from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, so 1825 cannot be called as a "cool" number.

Now, the question just asks of the total number of "cool" numbers, that may possibly exist.

Rohan took another number 1234 as an example. We quickly realised that 1 divides it, 2 divides it, but 3 does not, and there's a break there. So again 1234 is also not a "cool" number.

Next, he took another number 1240, again not divisible by 3, so again it did not work.

Then, Rohan used some of the ideas of divisibility. He said that the last digit of the number should be 5 or 0, for it to be divisible by 5, which I agreed.

Next, he told me to check the number 1260. This was a really good guess, so I will show what happens when 1260 is divided by all numbers from 1 to 10, one by one.

$$\rightarrow \frac{1260}{1} = 1260$$

$$\rightarrow \frac{1260}{2} = 630$$

$$\rightarrow \frac{1260}{3} = 420$$

$$\rightarrow \frac{1260}{4} = 315$$

$$\rightarrow \frac{1260}{5} = 252$$

$$\rightarrow \frac{1260}{6} = 210$$

$$\rightarrow \frac{1260}{7} = 180$$

$$\rightarrow \frac{1260}{8} = 157.5$$

$$\rightarrow \frac{1260}{9} = 140$$

$$\rightarrow \frac{1260}{10} = 126$$

So Close! Only 8 is the number which does not divide 1260, and hence the only obstacle, restricting it to become a "cool" number. However, given that we have got to a step like this, there is just a small clever way of which, if noticed, we can immediately can get a "cool" number from here.

Exercise:- With just some logic, can you get a "cool" number from here?

Hint:- What should you do to make this divisible by 8?

At first, Rohan was claiming that 8 can divide 1260. He had said that according to its divisibility rules, if the 3rd last digit of a number is even, then the number would be divisible by 8.

I provided him with a counterexample, 237 is a number not divisible by 8, but the 3rd last digit is even.

Then he thought for some time, and then claimed that if all the last 3 digits of the number is even, then it would work.

Again, I provided with another counterexample, 220 is a number where all the digits are even, but it is not divisible by 8.

So, he probably confused the exact statement of the divisibility rule of 8 with something else.

So I reminded him that the divisibility rules of 8 just states that, if the last 3 digits of the number itself, is divisible by 8, then so does the whole number. It has nothing to do with odd and even digits, it is just that the last 3 digits should be itself, divisible by 8.

And moreover, we have 4-digit numbers here, so divisibility rule of 8 will not be that much useful.

Next, Rohan took another 4-digit number, 1280. This immediately breaks, because 3 does not divide it.

At this point, I asked if he would continuously check numbers like this, or get a systematic method to solve the problem. Rohan chose the latter, and thought for some moment.

After about 2 minutes, Rohan said that we should factor out the number and check it.

He was exactly right, so I said, "Great, so if I factor out the number, then we would get factors of $2, 3, 5, 7$ and so on. Now what should I do to get a number divisible by each of the nos. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?"

I continued, "As 1 divides all numbers, we need not check it. So what should I do to make the numbers $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ divide that number?"

Rohan said, "So to make it divisible by 2, the number must be even, so there should be a factor of 2."

"Exactly", I replied. "So if there is a factor of 2, it will be divisible by 2. Now if there is just 1 factor of 2, will the number be divisible by 4?"

"No" was the answer from Rohan.

"Great, so what should I do to make it divisible by 4?"

Rohan said, "By taking 2 factors of 2."

And he is correct, so we finally got the idea to proceed. Similarly, to make it divisible by 8, we need no such divisibility rules, but just take 3 factors of 2, and that would make it divisible by $\{2, 4, 8\}$. Rohan agreed with this.

Next, I asked, "What should I do to make this number divisible by 3 now?"

At this point Rohan said to use the divisibility rules of 3, that the no. would be divisible by 3 if only its sum of its digits is divisible by 3, and he said that we could use it here.

He was correct about the rule, but I told him that here we are just finding out the number itself. The number is not known to us, and so are its digits, so how can we apply the rule to get something useful? We do not even know what the digits are.

I said, "Instead of thinking about the rule, if we just get a factor of 3 with the 3 factors of 2, then the number will be divisible by 3 too, isn't it?"

Rohan agreed with it. Next, since the number is divisible by $2 * 2 * 2 * 3$, it will also be divisible by $6 * 2 * 2$, so it will also be divisible by 6.

So we got the number to be fully divisible by $2, 3, 4, 6, 8$. Then I asked, "How should we continue to make the remaining numbers divide it?"

Rohan started with 5, "For the number 5, by its divisibility rule, the last digit should be 0 or 5."

I said, "Okay, that is a great claim, but here again, even if the last digit is known to us, the first 3 digits are still not known, so divisibility rule of 5 still does not help here."

Then Rohan said to take a factor of 5 to make it divisible by 5.

And indeed, he was correct. So it was evident at this point that thinking this problem in terms of factors was more useful than using the divisibility rules, particularly because the "cool" numbers are not known to us.

So I continued, "Hence the number must be of the form $2 * 2 * 2 * 3 * 5$.

Next, as Rohan also immediately spotted out, is that we can write this out as $2 * 2 * 3 * 10$, so 10 automatically divides this number.

Rohan said that only 7 and 9 are left now. For 9, Rohan suggested to put a 9 as a factor, to make it $2 * 2 * 2 * 3 * 5 * 9$, but I reminded that $9 = 3 * 3$, and we already had a 3, so instead, putting just a 3 also works!

So, we can make out that $2 * 2 * 2 * 3 * 3 * 5$ is divisible by 9. And now only 7 is left.

Since 7 cannot be divided like 9 (because 7 is a prime), we just have to take a factor of 7, and multiply it back, and that completes the number!

So, we finally have got $(2 * 2 * 2 * 3 * 3 * 5 * 7) = (2^3 * 3^2 * 5 * 7)$, and this number is indeed, divisible by all the numbers from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. A quick multiplication gives us the number to be 2520.

So, we got our first "cool" number 2520. Now, the question wants us to find all such "cool" numbers. So I asked Rohan what to do right now to find all such numbers.

Rohan immediately said of checking 5040, and indeed, he got the correct idea. For finding other "cool" numbers, they also should be divisible by the numbers from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, hence they should just be divisible by 2520. As 5040 is just the next multiple of 2520, which is just $2 * 2520$, just an extra factor of 2 is included, and it will not change the divisibility conditions of the original numbers. So 5040 will also be a "cool" number.

Then I asked him if there are more "cool" numbers. Rohan answered 10080, and it indeed is a "cool" number, but I quickly reminded him that the question only wanted of 4-digit "cool" numbers. So this number will not work.

Then Rohan said that only 2 such "cool" numbers exist. I told him that he was just... missing 1 number.

10080 was obtained when I did $(2520 * 4)$, and that is when it is becoming a 5 digit number. So multiplying higher numbers will not work.

But, we still didn't check $(2520 * 3)$. And Indeed, $(2520 * 3) = 7560$, which is both a 4 digit number and, by the same argument above, divisible by all the nos. from $\{1, 2, \ldots, 10\}$. Hence, 7560 is also a "cool" number.

Hence, there are 3 "cool" numbers, and that was our final answer.

Note:- This way of finding the "cool" numbers is actually a pretty known topic. In fact, the smallest "cool" number that we were looking for (which is 2520), is just the Lowest Common Multiple, or the LCM of the nos. $\{1, 2, \ldots, 10\}$. And the whole work that we did for this problem is nothing other than finding out the LCM, by factoring it.

Although I didn't mention it in the session, and Rohan probably knows about finding HCFs and LCMs before, doing the problem in a step-by-step way like this, brought out many other different and nice discussions which would have been missed had I just directly reminded him that it was just an LCM Problem.

For Instance, the fact that the Divisibility Rules of the numbers are not that much useful here, just because the numbers are not known to us, is a very nice point to take from here. So hopefully this discussion was helpful.

So that was all which we had discussed in the 9th Session. We had discussed 3 of the 4 problems which we had in the problem set, where I misinterpreted the first problem a bit, but still it was a nice discussion. And the 2nd problem was not fully discussed, since it required some advanced topics. So we had a discussion only up to the point when Rohan could understand it. And, I have tried to make this session as interesting, informative, and as enjoyable as possible.

Thank You For Reading!!

A TRIP BACK TO DIARIES

Souradip Das

April 13, 2023

Session 10

About Myself: Hello! I am Souradip Das, a student of class 10. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in the 9th session.

1 Introduction

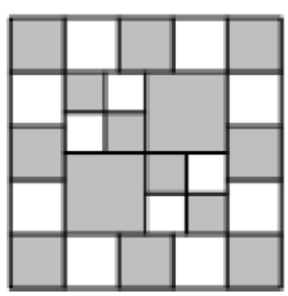
Note:- For my Board Exams this year, I was 4 months busy for its preparation so I couldn't get time to start diaries, although I did took math circles sessions fairly regularly. Thus I wish to start my 10th record with this session.

We had 1 student in the session, Raghav. He was a 7th grader.

We had discussed a total of 7 problems, 5 of which were given by cheenta, and the other 2 were made by me. I hope this Session will cover all the discussions that we had together.

2 Question 1

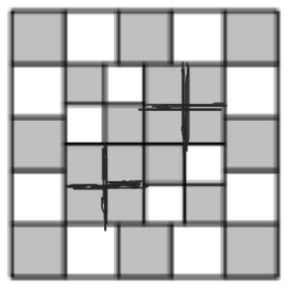
We started with the 1st Question. The diagram shows a large square divided into squares of three diferent sizes. What percentage of the large square is shaded?



This problem was fairly easy for a 7th grader, so I gave some time to Raghav to solve it. After some time, Raghav told me the answer was 68%. Since I hadn't solved it yet, from here we started the discussion.

Raghav's idea was that the figure could be considered as a 5×5 square, and there are 16 squares on the boundary, and only 8 of them are shaded. So exactly $\frac{1}{2}$ of the boundary squares is shaded.

Then, in the inner portion, we can divide them in this way and get 16 squares: Then exactly 12 of these smaller squares are shaded. So, exactly $\frac{3}{4}$ of

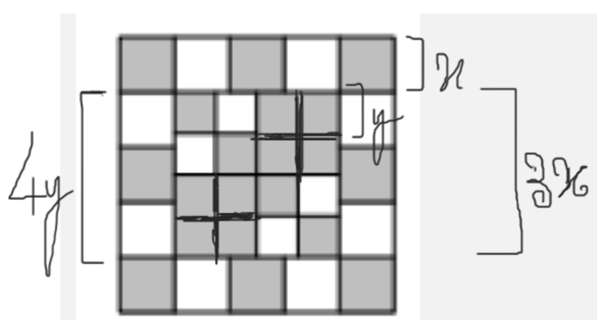


the inner squares is shaded.

After this, I asked him what to do next. He was thinking and that time and couldn't find a good way to proceed further.

Here, I suggested him to take variables for the sides of the different squares that he could see, say let the side of the smaller square be x, and that of the larger one be y. He agreed to me.

Then, he almost immediately figured out an useful expression and replied that $3x = 4y$. . This was absolutely correct, and it comes since the sides of 4 smaller squares coincide perfectly with the sides of 3 larger squares:



Now, it was pretty simple. The total area of the figure was just $(16x^2+16y^2)$, and the shaded area was $(8x^2 + 12y^2)$. So we were looking for the value of the ratio:

$$\left(\frac{8x^2+12y^2}{16x^2+16y^2}\right)$$

At this point, Raghav told me of changing the denominator of $(16x^2+16y^2)$ directly to $25x^2$ (since from his point of view, $25x^2$ was just the area of the full square), but it wasn't necessary, because it directly came from the relation $3x = 4y$, that we had found earlier.

So instead, I took $y = \frac{3x}{4}$ and substituted it back into the expression:

$$\left(\frac{8x^2+12y^2}{16x^2+16y^2}\right) = \left(\frac{8x^2+12\left(\frac{9x^2}{16}\right)}{16x^2+16\left(\frac{9x^2}{16}\right)}\right)$$

$$\implies \left(\frac{\frac{128x^2+108x^2}{16}}{\frac{256x^2+144x^2}{16}}\right)$$

$$\implies \frac{236x^2}{400x^2}$$

$$\implies \frac{59}{100}$$

So the answer was coming 59%. Raghav was still getting 68% somehow as his answer, so I double-checked myself and gave him some more time to check his answer again. And the answer was indeed 59%.

3 Question 2

We next moved to an easy problem.
The result of the calculation $(9 \times 11 \times 13 \times 15 \times 17)$ is the 6 - digit number $\overline{3n8185}$. What is the value of n?

I gave Raghav 5 minutes to solve this, and as expected, he quickly got the answer.

He just used the divisibility of 9 rule, and as 9 divides this number, 9 should also divide the sum of digits of the number. Hence, $9 \mid (25+n)$ and with $0 \leq n \leq 9$, the only possible value is $n = 2$. This was a very short problem.

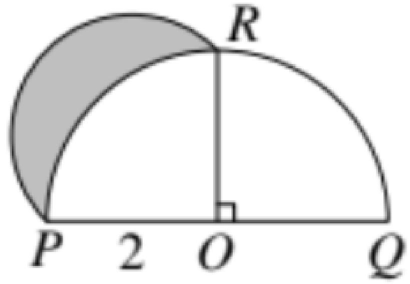
4 Question 3

Next we moved to a slightly harder but still an easy problem.
The positive integers m and n are such that $10 \times 2^m = 2^n + 2^{n+2}$. What is the difference between m and n?
Raghav took some time and proceeded in the correct way. He took 2^n common from the R.H.S. and this changes to :-
$\implies 10 \times 2^m = 2^n(1+4) = 2^n(5)$.

And from here it's very obvious that a factor of 5 can be cancelled both sides which gives :-
$\implies 2 \times 2^m = 2^{m+1} = 2^n$.
$\implies (m+1) = n$.
$\implies (m-n) = -1$.
And this is our required answer.

5 Question 4

Next we moved on to a moderate problem.
The diagram below, shows a semicircle with centre O and radius 2 and a semi-circular arc with diameter PR. Given $\angle POR = 90°$, find the area of the shaded region.



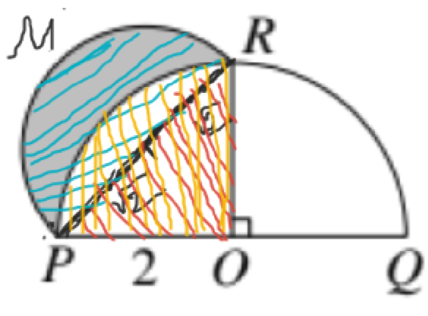
To start with, I joined PR and labelled some of the lengths. Raghav too quickly found $PR = 2\sqrt{2}$ from the Pythagoras Theorem.
Since the semi-circular arc is actually a semi-circle with diameter PR, it's radius would simply be $\sqrt{2}$.
Now, Raghav claimed that we should subtract the area of the triangle, from the area of the semi-circle of radius $sqrt2$. This actually dosen't work, and he quickly realized it.
Then he correctly stated that we should add the areas of the both the semi-circle and the triangle, then subtract the area of a quarter circle.

This works, because (Area of blue) + (Area of red) - (Area of yellow) = (Area of shaded region). (See Figure in the next Page)
Thus, our required answer is :-

$$\frac{1}{2}\pi(\sqrt{2})^2 + \frac{1}{2}(2)(2) - \frac{1}{4}\pi(2^2)$$

$$\implies \pi + 2 - \pi = 2.$$

Raghav was making some mistake in his calculations and was getting $(\pi + 2)$ as



his answer, but he he quickly got it right.
Another interesting fact related to this problem, was that the curved shaded region is the smallest lune (as a shape) that actually equals the area of a polygon (in this case, the triangle). There has been a numberphile video on this. It can be checked here: Lunes - Numberphile .

6 Question 5

This was the final problem from our paper.
Sam writes on a white board the positive integers from 1 to 6 inclusive, once each. She then writes p additional fives and q sevens on the board. The mean of all the numbers on the board is then 5.3. What is the smallest possible value of q?

After giving Raghav some time, he said that the sum of numbers from 1 to 6 was just $\frac{(6)(7)}{2} = 21$, then we add a total of p more fives and q more sevens. So the total sum adds up to $(21 + 5p + 7q)$.
And the total number of numbers was $(6 + p + q)$.
Thus we had the relation:

$$\frac{(21 + 5p + 7q)}{(6 + p + q)} = 5.3$$

$$\implies (21 + 5p + 7q) = (5.3)(6 + p + q)$$

Raghav suggested to multiply by 10 both sides to get rid of the decimal point, so I did that:

$$\implies (210 + 50p + 70q) = 53(6 + p + q)$$

$$\implies (210 + 50p + 70q) = 318 + 53p + 53q$$

$$\implies 17q = 3p + 108$$

After here, Raghav said to take a factor of 3 common from the R.H.S to give: $17q = 3(p + 36)$.
Then he argued that since q is an integer, we can claim that $3 \mid 17q$, or basically $3 \mid q$. And this was correct.
And then we could basically check out all multiples of 3 in the value of q, from ($q = 3, 6, 9, \dots$), till we get a positive integer solution for p.
Raghav did till here, and then in the end, found the smallest value of q to be 9, which is the correct answer.
Exercise:- Could you reason out why $q \leq 3, 6$ fails? Why does $q = 9$ work?

After this, we had completed solving all 5 problems that we had been provided from cheenta. Since we had some more time left, I decided to continue to discussion by giving some of my own made problems.

7 Question 6

This was a moderate but (I hope) a nice problem.
Let A, B, C, D be distinct digits from 0 to 9 (A, B, C are non-zero), such that:

$$\begin{array}{cccc} & A & B & A \\ \times & & & B \\ \hline C & C & D & D \end{array}$$

Given this has a unique solution, find it.
Raghav started the problem by converting the column wise expression into an equation:

$$(101A + 10B)B = (1100C + 11D)$$

Then he noticed a factor of 11 on the R.H.S., so he took a factor of 11 there:

$$(101A + 10B)B = 11(100C + D)$$

Then exactly like we had done in the previous problem, where we could claim that $3 \mid 17q$, since here too we have A, B to be positive integers, we can claim that $11 \mid (101A + B)B$.
But, as B is a single digit number, $11 \nmid B$. Thus, we have that $11 \mid (101A+10B)$.
Then, Raghav immediately started using modulo 11. (Which was the best approach here).
We have that $101A \equiv 2A \pmod{11}$, because $101 = (11)(9) + 2$.
Similarly, we could say that $10B \equiv -B \pmod{11}$.
Thus, we can say that $11 \mid (2A - B)$.
At this point, Raghav said that $(A, B) = (6, 1)$ is a solution. He was partly right, because this set may not necessarily give solutions for C and D.

Moreover, as $11 \mid (2A - B)$, $(2A - B)$ can be any multiple of 11, starting from $(0, 11, 22, \dots)$. But as $0 < A, B \leq 9$, the only possible options are:
(i) $(2A - B) = 0$
(ii) $(2A - B) = 11$
For the first case, Raghav claimed that $A < 5$ and $A > 0$. Or basically, $0 < A < 4$ and there are 4 choices for A. Checking all those choices gives us 4 solutions, $(A, B) = (1, 2), (2, 4), (3, 6), (4, 8)$.
Now these were only valid solutions for (A, B), we had to put them back in the equation to check if it provides valid solutions for C and D.
A quick checking gives us that none of these are valid solutions. So we move to the second case.
To get $(2A - B) = 11$, Raghav quickly listed the valid cases, which are $(A, B) = (6, 1), (7, 3), (8, 5), (9, 7)$.
Again a quick checking gives us that $(A, B) = (7, 3)$ indeed works, and this is the solution with valid values for C and D:

$$\begin{array}{cccc} & 7 & 3 & 7 \\ \times & & & 3 \\ \hline 2 & 2 & 1 & 1 \end{array}$$

Since there only existed a unique solution (as I had made the problem, I had checked all cases before and only this gave a solution), we stopped here and moved on to the next problem.

Exercise:- Since Raghav chose to attempt this problem as an equation form, he missed a very easier approach which could have avoided the use of mod 11. Could you find it? (**Hint**: Divisibility Rule of 11)

8 Question 7

Next I gave him an easier problem but with a little twist at the ending.
Consider a polygon whose smallest angle is $30°$, and all the angles are consecutive multiples of $30°$. What is the maximum number of sides this polygon can have?
Since we were almost out of time, I decided to give the main idea to start.
At this point Raghav had already figured out that $n = 3$ works (for the case of a right triangle), but since I asked for maximum, he started thinking again.
Let the number of possible sides of the polygon be n.
So this polygon has its angles in the form $(30°, 60°, \cdots, 30n°)$.
Now, we add the angles. There is a well-known formula for the sum of angles of a polygon, which is $180(n - 2)$.
Thus, we had:

$$30 + 60 + \cdots + 30n = 180(n - 2)$$

We took a factor of 30 common both sides, and cancelled them, which gave us:

$$1 + 2 + \cdots n = 6(n - 2)$$

Raghav was already aware of the famous formula $(1+2+\cdots+n) = \frac{n(n+1)}{2}$ (this was also used once in the 5th problem), so we used it here again:

$$\frac{n(n+1)}{2} = 6(n-2)$$

.

$$\implies n^2 + n = 12(n-2)$$

.

$$\implies n^2 - 11n + 24 = 0$$

.
Now a small factorization produced 2 roots, which are the only probable solutions to n. (He was also aware that $n = 3$ works, and quickly found the other root)

$$\implies (n-3)(n-8) = 0$$

Thus the roots are $3, 8$.
Raghav quickly claimed that the maximum number of possible sides to the polygon should be 8. But much to his surprise, and as it really should have been, the answer is not 8, but 3 only.
To find the reason, I gave Raghav some time to think. He thought that reflex angles occur for $n = 8$, which he is of course right, but they do not create much of a problem, as I reminded him that concave polygons too (those that have reflex angles) can exist. The problem was into something else.
Although I told Raghav the reason because the class had almost ended (he did say the idea was pretty interesting and it had a different way of thinking), I will not disclose the reason here, and let you all, the readers, try finding it.
Exercise:- Can you find the reason why $n = 8$ fails as a solution?

So that was all which we had discussed in the 10th Session. We had discussed 5 problems which we had in the problem set, and 2 additional problems that I had made myself. And, I have tried to make this session as interesting, informative, and as enjoyable as possible.

Thank You For Reading!!