Cheenta

IOQM Concepts Revisited

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Topic : Symmedians

Overview

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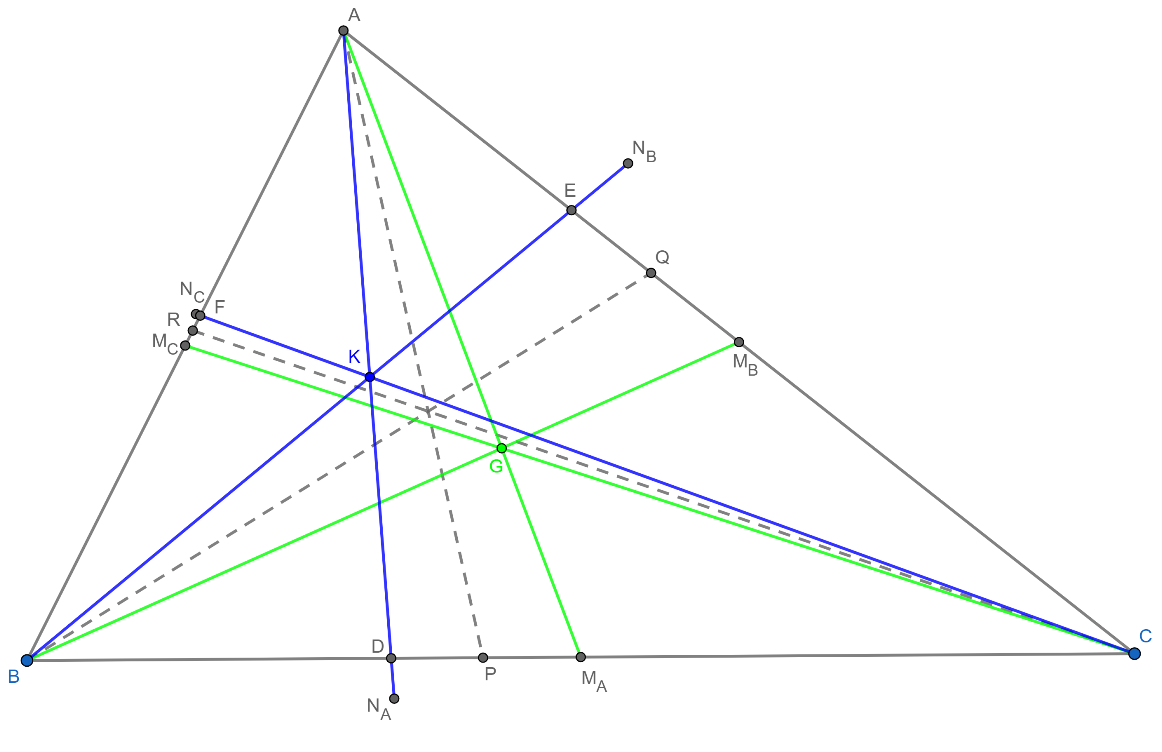
1 Problem statement with diagram

Let the $\triangle ABC$ have $AB = 7,\ BC = 11,\ CA = 10$. Let M_A, M_B, M_C be midpoints of $\overline{BC}, \overline{AC}$, and $\overline{AB}$ respectively. Let the angle bisectors of A, B and C intersect $\overline{BC}, \overline{AC}$, and $\overline{AB}$ at P, Q, and R, respectively. Reflect M_A about $\overline{AP}$, M_B about $\overline{BQ}$ and M_C about $\overline{CR}$ to obtain N_A, N_B, N_C, respectively. The lines $\overline{AN_A}, \overline{BN_B}$, and $\overline{CN_C}$ will then intersect $\overline{BC}, \overline{AC}$, and $\overline{AB}$ at D, E, and F respectively. Given that $\overline{BE}$, $\overline{CF}$ intersect at K inside the triangle, then find the ratio $\frac{[KAB]}{[KAC]}$, where $[XYZ]$ represents area of the $\triangle XYZ$.



Here, *dotted lines* represent *internal angle bisector*, *green lines* represent *median*, *blue lines* represent the lines joining A, N_A; B, N_B; C, N_C. Notice that the *dotted lines* concur at the *Incentre*, *green lines* concur at the *Centroid* of the $\triangle ABC$. Magically, the *blue lines* also concur, which is called the "*Symmedian point*" of the $\triangle ABC$. We will now learn about Symmedian and its properties.

2 Symmedian

Symmedian of a $\triangle ABC$ is the reflection of median to any side about angle bisector of the angle opposite to that side. Given below is an illustration of A - symmedian,

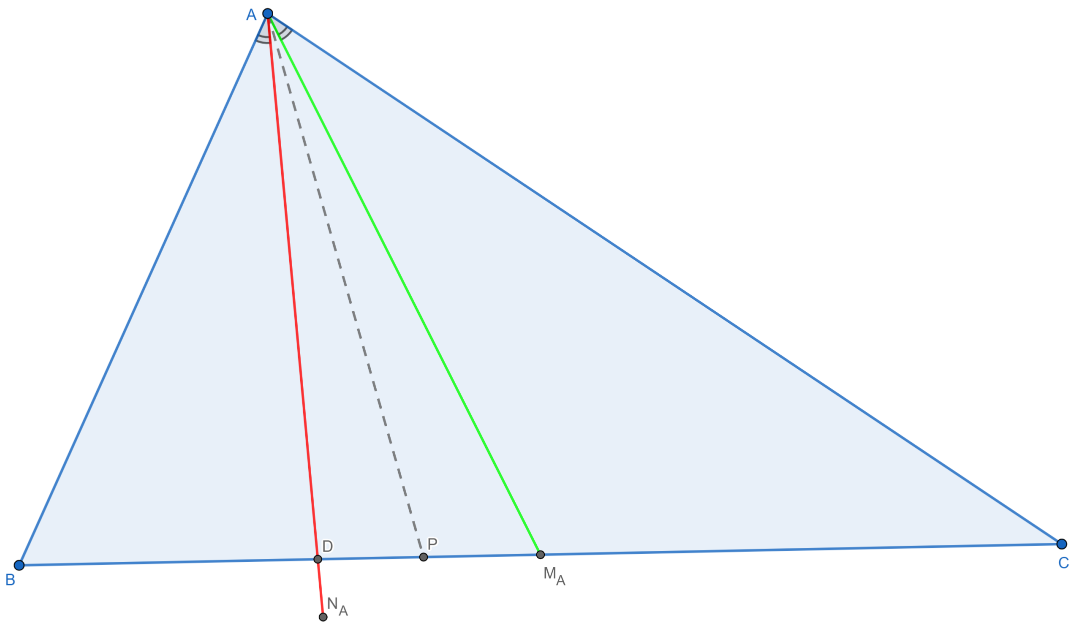


Figure 1: A - Symmedian

The notations have usual meaning as given in the Section 1 (N_A is the reflection of M_A about the angle bisector AP). The red line is the A- symmedian. Notice that since it is a reflection, $\angle N_AAP = \angle M_AAP$ and $\angle BAP = \angle CAP = \frac{\angle A}{2} \Rightarrow \angle BAD = \angle CAM_A$.

So, the symmedian make angles with the adjacent sides that are equal to the angles made by the median with the same adjacent sides but in the opposite orientation. Lines that satisfy the above condition are called as *isogonal lines*, which are obtained by reflecting about corresponding angle bisector.

Before proceeding, let us find $\frac{\sin\angle CAM_A}{\sin\angle BAM_A}$,

$$\begin{aligned}
1 = \frac{CM_A}{BM_A} &= \frac{[CAM_A]}{[BAM_A]} \text{ (since they share common height)} \\
\Rightarrow 1 &= \frac{\frac{1}{2}\times AC\times AM_A\times \sin\angle CAM_A}{\frac{1}{2}\times AB\times AM_A\times \sin\angle BAM_A} \\
\Rightarrow 1 &= \frac{AC\sin\angle CAM_A}{AB\sin\angle BAM_A} \\
\Rightarrow \frac{AB}{AC} &= \frac{\sin\angle CAM_A}{\sin\angle BAM_A} \qquad (2.1)
\end{aligned}$$

Now, let us find the ratio $\frac{BD}{CD}$, inorder to identify the position of foot of the $A-$ symmedian,

$$\begin{aligned}
\frac{BD}{CD} &= \frac{[BAD]}{[CAD]} \text{ (since they share common height)} \\
&= \frac{\frac{1}{2}\times AB\times AD\times \sin\angle BAD}{\frac{1}{2}\times AC\times AD\times \sin\angle CAD} \\
&= \frac{AB\sin\angle CAM_A}{AC\sin\angle BAM_A} \text{ (since median and symmedian are isogonal)} \\
\Rightarrow & \boxed{\frac{BD}{CD} = \left(\frac{AB}{AC}\right)^2} \text{ (from the Equation 2.1)}
\end{aligned}$$

Suppose if E, F are feet of $B, C-$ symmedians respectively, then by similar approach we get,

$$\boxed{\frac{BD}{CD} = \left(\frac{AB}{AC}\right)^2 \ ; \ \frac{CE}{AE} = \left(\frac{BC}{BA}\right)^2 \ ; \ \frac{AF}{BF} = \left(\frac{CA}{CB}\right)^2} \qquad (2.2)$$

Note that AD, BE, CF are symmedians respectively if and only if the points D, E, F divide BC, CA, AB respectively, in ratio given in the Equation 2.2.

3 Symmedian point

Let us now prove that all three symmedians concur at a point inside the $\triangle ABC$ called the "*Symmedian point*" of the triangle. Before that let us learn about the *Ceva's Theorem*. The *Ceva's Theorem* states that,

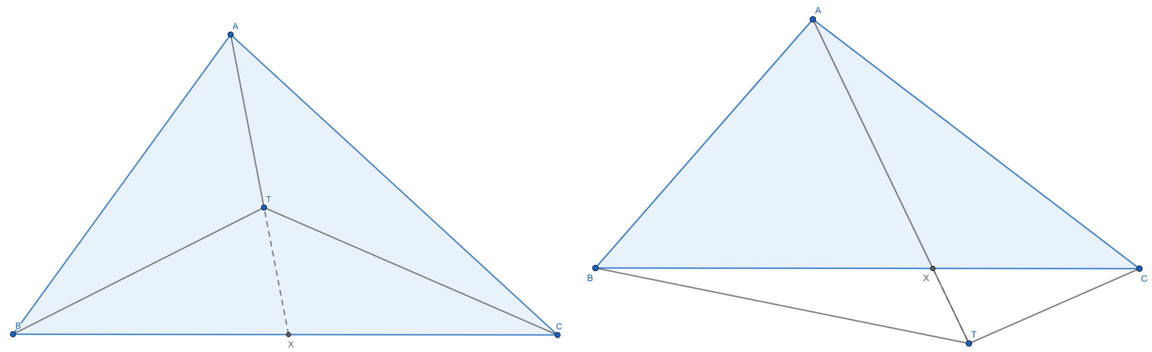
> Consider points $X, Y, Z (\neq A, B, C)$ on lines BC, CA, AB respectively of a $\triangle ABC$. Then the cevians AX, BY, CZ concur at a point (inside (or) outisde the triangle) if and only if $\frac{BX}{CX} \times \frac{CY}{AY} \times \frac{AZ}{BZ} = 1$ with *odd* (3 (or) 1) number of points out of X, Y, Z are contained inside the sides of the triangle.

In the next section we will discuss an elegant proof of the ceva's theorem. Now, let us take $X = D,\ Y = E,\ Z = F$ which are the feet of symmedians. Then by the Equation 2.2,

$$\frac{BD}{CD} \times \frac{CE}{AE} \times \frac{AF}{BF} = \left(\frac{AB}{AC}\right)^2 \times \left(\frac{BC}{BA}\right)^2 \times \left(\frac{CA}{CB}\right)^2 = 1$$

All three (*odd*) feet of symmedians are inside the triangle as the median is inside the triangle. Hence, we see that the symmedians concur at the "*Symmedian point*". This point must be inside the triangle.

4 Area ratios



(a) T above side BC

(b) T below side BC

Figure 2: All configurations of T w.r.t $\triangle ABC$

Consider a $\triangle ABC$ with a point T (not on AB, AC) interior (or) exterior of it such that AT is not parallel to BC and intersects BC at X. Now, let us find $\frac{[ATB]}{[ATC]}$.

$$\frac{BX}{CX} = \frac{[AXB]}{[AXC]} \text{ (since they share common height)}$$
$$\frac{BX}{CX} = \frac{[TXB]}{[TXC]} \text{ (since they share common height)}$$

By the Componendo - Dividendo rule,

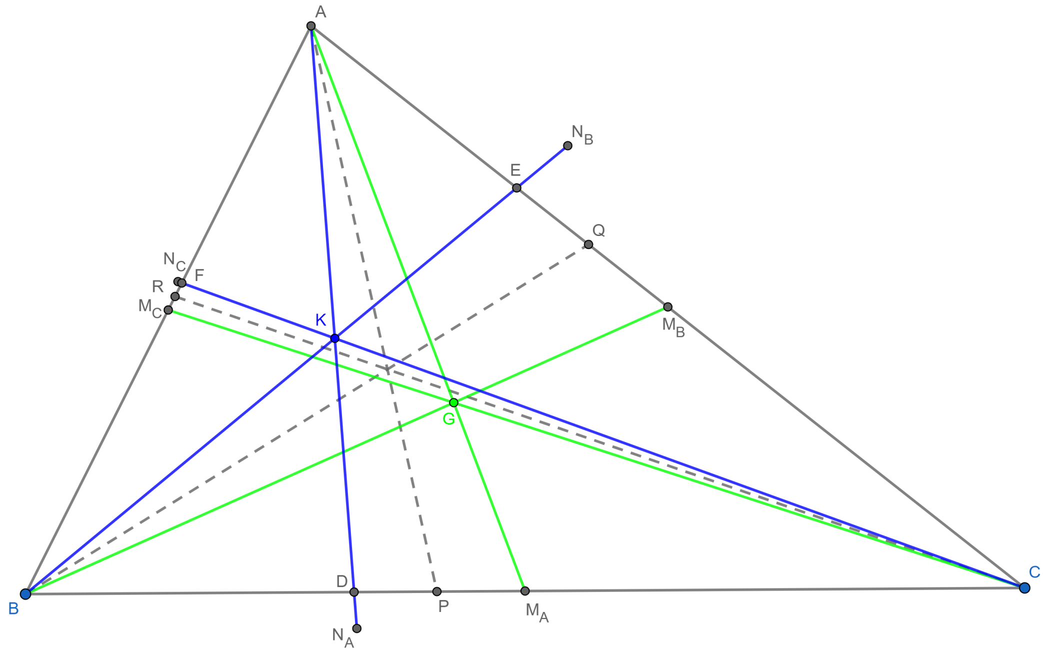
$$\Rightarrow \frac{BX}{CX} = \frac{[AXB]+[TXB]}{[AXC]+[TXC]} \text{ (for the Figure 2b)}$$
$$= \frac{[AXB]-[TXB]}{[AXC]-[TXC]} \text{ (for the Figure 2a)}$$
$$\Rightarrow \boxed{\frac{BX}{CX} = \frac{[ATB]}{[ATC]}} \text{ (For both cases in the Figure 2)} \tag{4.1}$$

The above result also proves the "only if" condition of the *Ceva's Theorem* because suppose the cevians AX, BY, CZ given in the Section 3 concur at T,

$$\Rightarrow \frac{BX}{CX} \times \frac{CY}{AY} \times \frac{AZ}{BZ} = \frac{[ATB]}{[ATC]} \times \frac{[BTC]}{[ATB]} \times \frac{[ATC]}{[BTC]} = 1$$

But for the "if", i.e., converse of the condition, we will use *Phantom point* method by considering CT to intersect AB at Z', where T is the point of intersection of AX, BY. Now, by the "only if" condition we have, $\frac{AZ}{BZ} = \frac{AZ'}{BZ'} \Rightarrow Z = Z'$ because we know the fact whether Z, Z' lies inside (or) outside the side as *odd* number of points must be inside the sides of $\triangle ABC$.

5 Solution to the Problem statement



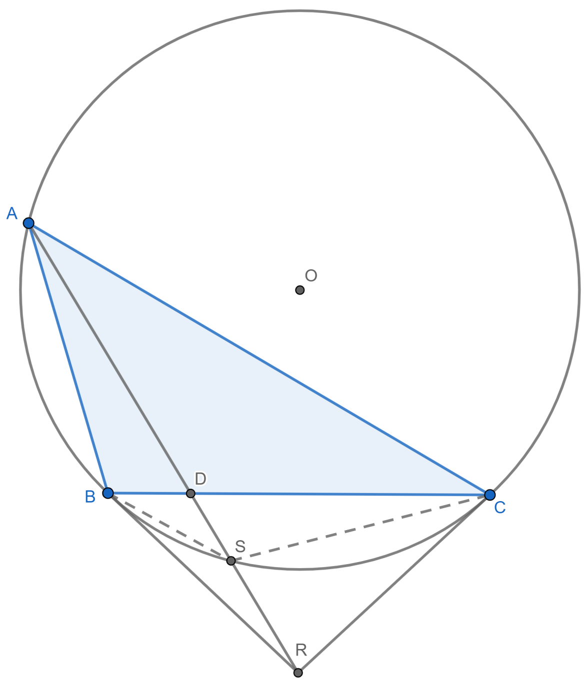
Let us now come back to the original problem statement. It says that M_A is reflected about the angle bisector to N_A and we join AN_A, which will be the A-Symmedian because 2 points define a line. Similarly for, BN_B, CN_C and hence all three concur at K, which is the "*Symmedian point*". Now, by the Equations 2.2, 4.1, we get, $\frac{[KAB]}{[KAC]} = \frac{BD}{CD} = \left(\frac{AB}{AC}\right)^2 = \left(\frac{7}{10}\right)^2 \boxed{= \frac{49}{100}}$.

6 Construction of Symmedian with ruler and compass

Suppose you are given a $\triangle ABC$ and we will now construct the $A-$ symmedian of the triangle. A similar contruction can also be done for $B, C-$ symmedians,

6.1 Rules for construction

1. Construct the *circumcentre* of $\triangle ABC$. To do this, draw perpendicular bisectors of any two sides with equal arc in compass at both endpoints of a side and on both upper and lower part of a side and join it by ruler. The two formed lines will intersect at the *circumcentre*, say O.
2. Draw the *circumcircle* Γ of $\triangle ABC$ with compass.
3. Draw the tangents to Γ at B, C by drawing a perpendicular line to OB, OC at B, C respectively. Let the tangents intersect at R.
4. Join AR, which will be the $A-$ symmedian of the $\triangle ABC$.



6.2 Proof of construction

Here are some hints for the proof of symmedian construction above. Let AR intersect BC at D and Γ at S other than A, then

- $\frac{BD}{CD} = \frac{BS \sin C}{CS \sin B}$
- $\triangle BSR \sim \triangle ABR$; $\triangle CSR \sim ACR$
- $\frac{BS}{CS} = \frac{AB}{AC}$
- $\frac{BD}{CD} = \frac{AB \sin C}{AC \sin B} = \left(\frac{AB}{AC}\right)^2$
- Hence, D is the foot of $A-$ symmedian by the Equation 2.2

Question 6.1. *Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines BN and CM meet at point P. The circumcircles of triangles BMP and CNP meet at two distinct points P and Q. Then, prove that the line AQ is the A-symmedian of triangle ABC.*
[*Hint :* $\triangle BQM \sim \triangle NQC$]

Question 6.2. *Let ABC be a triangle with $AC = BC$, and P a point inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB, then show that $\angle APM + \angle BPC = 180°$.*